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LAGRANGE INVERSION OVER FINITE FIELDS

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A finite field analogue of the Lagrange inversion formula is given and applications to the derivation of character sum identities are discussed.

1. Introduction. In this paper we discuss character sum analogues for Lagrange inversion in one or several variables over finite fields. We then use these techniques as tools for deriving character sum identities. We begin with a short description of classical Lagrange inversion.

If $f(z)$ and $g(z)$ are formal power series where $g(0) = 0$ but $g'(0) \neq 0$, the inversion problem is to write f as a power series in the variable $g(z)$. The Lagrange inversion theorem gives the solution to this problem. Specifically, the result is

$$f(z) = f(0) + \sum_{k=1}^{\infty} c_k g(z)^k,$$

where

$$c_k = \operatorname*{Res}_{z} \frac{f'(z)}{kg(z)^k}, \tag{1.1}$$

or alternatively,

$$c_k = \operatorname*{Res}_{k} \frac{f(z)g'(z)}{g(z)^{k+1}}. \tag{1.2}$$

Simple proofs of these results can be found in [7] or [**10**]. These references also contain multivariable generalizations.

Recently there has been much work in developing q-analogues for Lagrange inversion, for example, see [**1**], [3] or [**4**]. Lagrange inversion is useful in special functions and comes up frequently in deriving transformations and summation theorems as in [**5**] or [**6**].

This paper is organized as follows. Theorems analogous to Lagrange inversion are derived in §2. The strengths and weaknesses of the analogy are discussed in §3. In §4 we derive several character sum identities from these theorems. We set notation in the remainder of this section.

Throughout this paper, $GF(q)$ is the finite field with $q = p^n$ elements, where p is an odd prime. The capital letters A, B, L, M, N and R and Greek letters χ and θ will denote arbitrary multiplicative characters of $GF(q)$. The quadratic character will be denoted by φ and the trivial character by ε. All multiplicative characters are defined to be 0 at the 0 element of $GF(q)$. Define $\overline{A}$ by $A\overline{A} = \varepsilon$. We define a function δ on $GF(q)$ by

$$\delta(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases} \tag{1.3}$$

and on multiplicative characters of $GF(q)$ by

$$\delta(A) = \begin{cases} 1, & \text{if } A = \varepsilon, \\ 0, & \text{if } A \neq \varepsilon. \end{cases} \tag{1.4}$$

Note that $\delta(x) = 1 - \varepsilon(x)$. Write $\sum_x$ to denote the sum over all $x \in GF(q)$ and $\sum_\chi$ to denote the sum over all multiplicative characters of $GF(q)$. Let $\zeta = e^{2\pi i/p}$ and set Tr equal to the trace map from $GF(q)$ to $GF(p)$.

We will make use of the orthogonality relations [9, pp. 89, 90]

$$\sum_x \chi(x) = (q-1)\delta(\chi), \tag{1.5}$$

$$\sum_\chi \chi(x) = (q-1)\delta(1-x), \tag{1.6}$$

and

$$\sum_x \zeta^{\mathrm{Tr}(xy)} = q\delta(y). \tag{1.7}$$

The Gauss sum of a multiplicative character A is defined by

$$G(A) = \sum_x A(x)\zeta^{\mathrm{Tr}(x)}, \tag{1.8}$$

and the Jacobi sum of A and B is defined by

$$J(A,B) = \sum_x A(x)B(1-x). \tag{1.9}$$

Finally, some easy changes of variables in (1.9) imply

$$J(A,B) = J(B,A) \tag{1.10}$$

and

$$J(A,B) = B(-1)J(\overline{AB}, B). \tag{1.11}$$

2. Inversion over finite fields. Suppose $\{f_1(x), \ldots, f_q(x)\}$ is an orthonormal basis for the vector space of all complex valued functions over $GF(q)$ with respect to the inner product

$$\langle f(x), g(x)\rangle = \sum_x f(x)\overline{g(x)}. \tag{2.1}$$

Thus,

$$\sum_x f_i(x)\overline{f_j(x)} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.2}$$

Let M be the $q \times q$ matrix $(f_i(x_j))_{ij}$. Then (2.2) implies $MM^* = I$, where M^* is the conjugate transpose of M. Consequently, $M^*M = I$ from which it follows that

$$\sum_{k=1}^{q} f_k(x)\overline{f_k(y)} = \delta(x - y). \tag{2.3}$$

If $f(x)$ is any function from $GF(q)$ to $\mathbf{C}$, the orthogonality relations (2.2) and (2.3) imply

$$f(x) = \sum_{k=1}^{q} c_k f_k(x), \tag{2.4}$$

where

$$c_k = \sum_x f(x)\overline{f_k(x)}. \tag{2.5}$$

LEMMA 2.6. *Let $f(x)$: $GF(q) \to \mathbf{C}$ and $g(x)$: $GF(q) \to GF(q)$. Then for fixed x,*

$$\sum f(y) = \sum_{k=1}^{q} c_k f_k(g(x)),$$

where

$$c_k = \sum_y f(y)\overline{f_k(g(y))},$$

and the sum on the left hand side extends over all y such that $g(y) = g(x)$.

Proof. With c_k defined as above we have

$$\begin{aligned} \sum_{k=1}^{q} c_k f_k(g(x)) &= \sum_{k=1}^{q} \sum_y f(y)\overline{f_k(g(y))} f_k(g(x)) \\ &= \sum_y f(y)\delta(g(x) - g(y)) = \sum_{\substack{y: \\ g(y)=g(x)}} f(y) \end{aligned}$$

as desired.

The two classical examples of orthogonal bases for complex valued functions over $GF(q)$ are the set of all multiplicative characters together with $\delta(x)$, and the set of all additive characters. That these are in fact orthogonal bases follows from (1.5) and (1.7). These sets can be made orthonormal by appropriate scaling so we have the following as corollaries to Lemma 2.6.

THEOREM 2.7. *Given $f(x)\colon GF(q) \to \mathbf{C}$ and $g(x)\colon GF(q) \to GF(q)$,*
(a)

$$\sum_{\substack{y:\\ g(y)=g(x)}} f(y) = \delta(g(x)) \sum_{\substack{y:\\ g(y)=0}} f(y) + \sum_{\chi} c_\chi \chi(g(x)),$$

where

$$c_\chi = (1/(q-1)) \sum_y f(y)\bar{\chi}(g(y)),$$

(b)

$$\sum_{\substack{y:\\ g(y)=g(x)}} f(y) = \sum_y c_y \zeta^{\mathrm{Tr}(g(x)y)},$$

where $c_y = (1/q)\sum_x f(x)\zeta^{-\mathrm{Tr}(g(x)y)}$.

The generalization to several variables causes no problems. For example, in the case of functions of two variables it is clear that arguments similar to those in Lemma 2.6 will show the following.

THEOREM 2.8. *If $f(x, y)\colon GF(q)^2 \to \mathbf{C}$ and $g(x, y)$, $h(x, y)\colon GF(q)^2 \to GF(q)$, then*

$$\varepsilon(g(x, y)h(x, y)) \sum f(u, v) = \sum_{\chi,\theta} c_{\chi,\theta} \chi(g(x, y))\theta(h(x, y)),$$

where

$$c_{\chi,\theta} = \frac{1}{(q-1)^2} \sum_{x,y} f(x, y)\bar{\chi}(g(x, y))\bar{\theta}(h(x, y))$$

and the sum on the left hand side extends over all (u, v) for which $g(u, v) = g(x, y)$ and $h(u, v) = h(x, y)$.

Of course, a similar result holds if the multiplicative characters in Theorem 2.8 are replaced by any orthogonal basis for the complex-valued functions over $GF(q)$.

3. Remarks. There are several drawbacks to Theorems 2.7 and 2.8. We will discuss them while making more explicit the analogy between Lagrange inversion and the results of §2.

The result in (1.2) can be restated

$$c_k = \text{C.T.}\, zf(z)g'(z)g(z)^{-1-k} \tag{3.1}$$

where C.T. $f(z)$ is the constant term in the Laurent expansion for $f(z)$. Note that if a function $f(x)$: $GF(q) \to \mathbf{C}$ is expanded as a character sum,

$$f(x) = f(0)\delta(x) + \sum_{\chi} c_{\chi}\chi(x), \tag{3.2}$$

then

$$c_{\varepsilon} = \frac{1}{q-1} \sum_{x \neq 0} f(x).$$

Inspired by this result, define the constant term of the function $f(x)$ by

$$\text{C.T.}\, f(x) = \frac{1}{q-1} \sum_{x \neq 0} f(x). \tag{3.3}$$

With this definition, the constants c_{χ} in Theorem 2.7(a) are defined by

$$c_{\chi} = \text{C.T.}\, f(x)\bar{\chi}(g(x)), \tag{3.4}$$

which we contrast with (3.1).

The conditions $g(0) = 0$, $g'(0) \neq 0$ in the classical theorem imply that $g(z)$ is one-to-one near $z = 0$. If $g(x)$: $GF(q) \to GF(q)$ is one-to-one, then the sum on the right hand side of Lemma 2.6 reduces to a single term so with c_k as in Lemma 2.6, $f(x) = \sum_{k=1}^{q} c_k f_k(g(x))$. Unfortunately, functions g: $GF(q) \to GF(q)$ which arise in practice tend not to be one-to-one. Consequently, for most practical problems, the summation on the left hand side of Lemma 2.6 is required.

In the classical theorem, the coefficients c_k are unique in the sense that if

$$\sum_{k=1}^{\infty} c_k g(z)^k = \sum_{k=1}^{\infty} d_k g(z)^k$$

in some neighborhood of $z = 0$, then $c_k = d_k$. The worst drawback to Lemma 2.7 is that the coefficients c_k are uniquely determined only if $g(x)$ is one-to-one. If $g(x)$ is not one-to-one, the best that can be said is that if

$$\sum_{k=1}^{q} c_k f_k(g(x)) = \sum_{k=1}^{q} d_k f_k(g(x)),$$

then $c_k = d_k + h_k$ where the coefficients h_k satisfy

$$\sum_{k=1}^{q} h_k f_k(g(x)) = 0,$$

for all x. For example, if $f(x) = \delta(1 - x)$ and $g(x) = x^2$, then Theorem 2.7(a) gives

$$\delta(1 - x) + \delta(1 + x) = \frac{1}{q - 1}\sum_{\chi}\chi(x^2). \tag{3.5}$$

If a is any element of $GF(q)$ which is not a square, then

$$\frac{1}{q - 1}\sum_{\chi}\chi(ax^2) = \delta(1 - ax^2) = 0$$

for all x. Thus we have

$$\frac{1}{q - 1}\sum_{\chi}\chi(x^2) = \frac{1}{q - 1}\sum_{\chi}(1 + \chi(a))\chi(x^2)$$

for all x in $GF(q)$. The consequence of these remarks is that the classical technique of expanding a function in terms of another function in two different ways and equating coefficients does not generalize well to finite fields.

4. Examples. In this section we give some short examples of Theorem 2.7 and an extended discussion of the uses of Theorem 2.8 in deriving character sum identities.

As a first example, if $f(x) = A(1 - x)$ and $g(x) = x$, then

$$c_\chi = \frac{1}{q - 1}\sum_{y} A(1 - y)\bar{\chi}(y) = \frac{1}{q - 1}J(A, \bar{\chi}).$$

By Theorem 2.7(a), we have

$$A(1 - x) = \delta(x) + \frac{1}{q - 1}\sum_{\chi} J(A, \bar{\chi})\chi(x). \tag{4.1}$$

This useful result is an analogue for the binomial theorem, which can be made more striking by the introduction of "binomial coefficients" (see [8])

$$\binom{A}{B} = \frac{B(-1)}{q}J(A, \bar{B}). \tag{4.2}$$

With this definition, (4.1) becomes

$$A(1 - x) = \delta(x) + \frac{q}{q - 1}\sum_{\chi}\binom{A}{\chi}\chi(-x). \tag{4.3}$$

From (1.11) we derive a variant of (4.1),

$$A(1 - x) = \delta(x) + \frac{1}{q - 1}\sum_{\chi} J(\bar{A}\chi, \bar{\chi})\chi(-x), \tag{4.4}$$

or, in terms of binomial coefficients,

$$A(1-x) = \delta(x) + \frac{q}{q-1}\sum_{\chi}\binom{\bar{A}\chi}{\chi}\chi(x). \tag{4.5}$$

Examples of the uses of (4.1) and (4.3) can be found in [8].

For an example of Theorem 2.7(b), let $f(x) = \zeta^{\mathrm{Tr}(2x)}$ and $g(x) = x^2$. Since $g(x) = g(-x)$ but $x \neq -x$ unless $x = 0$,

$$\sum_{g(y)=g(x)} f(x) = \zeta^{\mathrm{Tr}(2x)} + \zeta^{-\mathrm{Tr}(2x)} - \delta(x).$$

On the right hand side of Theorem 2.7(b) we must calculate

$$c_y = \frac{1}{q}\sum_x \zeta^{\mathrm{Tr}(2x - x^2 y)}.$$

Note that $c_0 = 0$. For $y \neq 0$, replace x by $x + 1/y$ to obtain

$$\begin{aligned} c_y &= \frac{1}{q}\sum_x \zeta^{\mathrm{Tr}(1/y - x^2 y)} = \frac{1}{q}\zeta^{\mathrm{Tr}(1/y)}\sum_x \zeta^{-\mathrm{Tr}(x^2 y)} \\ &= \frac{1}{q}\zeta^{\mathrm{Tr}(1/y)}\sum_x \zeta^{-\mathrm{Tr}(xy)}(1 + \varphi(x)) = \frac{1}{q}\zeta^{\mathrm{Tr}(1/y)}\sum_x \varphi(x)\zeta^{-\mathrm{Tr}(xy)} \\ &= \frac{1}{q}\zeta^{\mathrm{Tr}(1/y)} G(\varphi)\varphi(-y). \end{aligned}$$

By Theorem 2.7(b) we now have

$$\zeta^{\mathrm{Tr}(2x)} + \zeta^{-\mathrm{Tr}(2x)} - \delta(x) = \frac{1}{q}\varphi(-1)G(\varphi)\sum_y \varphi(y)\zeta^{\mathrm{Tr}(1/y + x^2 y)}. \tag{4.6}$$

Replacing y by $1/y$ and using $G(\varphi)^2 = q\varphi(-1)$ gives

$$\sum_y \varphi(y)\zeta^{\mathrm{Tr}(y + x^2/y)} = G(\varphi)\left(\zeta^{\mathrm{Tr}(2x)} + \zeta^{\mathrm{Tr}(-2x)} - \delta(x)\right). \tag{4.7}$$

This is a character sum analogue for

$$\int_0^\infty \sqrt{y}\, e^{-y - x^2/y}\,\frac{dy}{y} = \Gamma\left(\frac{1}{2}\right)e^{-2x}. \tag{4.8}$$

We now derive an identity from Theorem 2.8 and use it to obtain some more substantial character sum formulas. This example was inspired by [6] and follows that paper (by analogy) closely.

Consider functions $f(x, y)$ with the property that $f(x, y) = 0$ whenever $x = 0$, $y = 0$, $x = 1$, $y = 1$ or $xy = 1$. Let

$$g(x, y) = \frac{x(1-y)}{1-xy} \quad \text{and} \quad h(x, y) = \frac{(1-x)y}{1-xy}.$$

The mapping

$$(g(x, y), h(x, y)) \colon GF(q)^2 \to GF(q)^2$$

is one-to-one on the set of all (x, y) such that $xy(1 - x)(1 - y)(1 - xy) \neq 0$. Consequently, by Theorem 2.8 we have

$$f(x, y) = \sum_{\chi, \theta} c_{\chi, \theta} \chi(g(x, y)) \theta(h(x, y)),$$

where

$$c_{\chi, \theta} = \frac{1}{(q-1)^2} \sum_{u, v} f(u, v) \bar{\chi}(g(u, v)) \bar{\theta}(h(u, v)).$$

Also,

$$g\left(\frac{x}{1-y}, \frac{y}{1-x}\right) = x, \qquad h\left(\frac{x}{1-y}, \frac{y}{1-x}\right) = y,$$

so

$$f\left(\frac{x}{1-y}, \frac{y}{1-x}\right) = \sum_{\chi, \theta} c_{\chi, \theta} \chi(x) \theta(y).$$

If we denote

$$\frac{1}{(q-1)^2} \sum_{x, y} f(x, y) \quad \text{by C.T.} f(x, y),$$

then

$$\begin{aligned} \text{C.T.} f\left(\frac{x}{1-y}, \frac{y}{1-x}\right) &= c_{\varepsilon\varepsilon} \\ &= \frac{1}{(q-1)^2} \sum_{u, v} f(u, v) \varepsilon(uv(1-u)(1-v)(1-uv)) \\ &= \frac{1}{(q-1)^2} \sum_{u, v} f(u, v) = \text{C.T.} f(x, y). \end{aligned}$$

We have proved:

THEOREM 4.9. *If $f(x, y) \colon GF(q)^2 \to \mathbf{C}$ satisfies $f(x, y) = 0$ for all (x, y) such that $xy(1 - x)(1 - y)(1 - xy) = 0$, then*

$$\text{C.T.} f\left(\frac{x}{1-y}, \frac{y}{1-x}\right) = \text{C.T.} f(x, y).$$

This result is the analogue of the classical result **[6, (3)]**

$$\text{(4.10)} \qquad \text{C.T.} f\left(\frac{x}{1-y}, \frac{y}{1-x}\right) = \text{C.T.} \frac{1}{1-xy} f(x, y).$$

If we now take $f(x, y) = A(1 - x)B(1 - y)\bar{L}(x)\bar{M}(y)\overline{AB}(1 - xy)$, then

$$f\left(\frac{x}{1-y}, \frac{y}{1-x}\right) = \varepsilon(1 - x - y)AM(1 - x)BL(1 - y)\bar{L}(x)\bar{M}(y)$$

$$= AM(1 - x)BL(1 - y)\bar{L}(x)\bar{M}(y)$$

$$-\delta(1 - x - y)A(1 - x)B(x).$$

By Theorem 4.9 we have

$$(4.11)\qquad \sum_{x,y} A(1 - x)B(1 - y)\bar{L}(x)\bar{M}(y)\overline{AB}(1 - xy)$$

$$= \sum_{x,y} AM(1 - x)BL(1 - y)\bar{L}(x)\bar{M}(y)$$

$$- \sum_{x,y} \delta(1 - x - y)A(1 - x)B(x)$$

$$= J(AM, \bar{L})J(BL, \bar{M}) - J(A, B).$$

From (4.4) it follows that

$$(4.12)\qquad \overline{AB}(1 - xy) = \delta(xy) + \frac{1}{q-1}\sum_{\chi} J(AB\chi, \bar{\chi})\chi(-xy).$$

When (4.12) is substituted into the left hand side of (4.11) and the x and y sums are evaluated we have

$$(4.13)\qquad \frac{1}{q-1}\sum_{\chi} J(A, \bar{L}\chi)J(B, \bar{M}\chi)J(AB\chi, \bar{\chi})\chi(-1)$$

$$= J(AM, \bar{L})J(BL, \bar{M}) - J(A, B).$$

In terms of binomial coefficients,

$$(4.14)\qquad \frac{q}{q-1}\sum_{\chi}\binom{A}{L\bar{\chi}}\binom{B}{M\bar{\chi}}\binom{AB\chi}{\chi} = \binom{AM}{L}\binom{BL}{M} - \frac{BML(-1)}{q}\binom{A}{\bar{B}}.$$

This is an analogue for the binomial coefficient version of Saalschütz's theorem [**6**, (1)]

$$(4.15)\qquad \sum_{k\geq 0}\binom{a}{l-k}\binom{b}{m-k}\binom{a+b-k}{k} = \binom{a+m}{l}\binom{b+l}{m}.$$

If we take

$$f(x, y) = \varepsilon(1 - x)\varepsilon(1 - y)\bar{L}(x)\bar{M}(y)N(x - y)\bar{N}(1 - xy),$$

then Theorem 4.9 gives

$$\text{(4.16)}\quad \text{C.T.}\big(\varepsilon(1-x)\varepsilon(1-y)\overline{L}(x)\overline{M}(y)N(x-y)\overline{N}(1-xy)\big)$$
$$= \text{C.T.}\big(\overline{L}(x)M(1-x)\overline{M}(y)L(1-y)N(x-y)\varepsilon(1-x-y)\big).$$

Using $\varepsilon(a) = 1 - \delta(a)$ and converting (4.16) to a sum we obtain

$$\text{(4.17)}\quad \sum_{x,y} \overline{L}(x)M(1-x)\overline{M}(y)L(1-y)N(x-y) - (q-1)\delta(N)$$
$$= \sum_{x,y} \overline{L}(x)\overline{M}(y)N(x-y)\overline{N}(1-xy)$$
$$-(q-1)\delta(M) - (q-1)N(-1)\delta(L).$$

Writing $N(x-y) = N(x)N(1-y/x)$ and applying (4.1) to $N(1-y/x)$ and $\overline{N}(1-xy)$, the summation on the right hand side becomes

$$\frac{1}{(q-1)^2} \sum_{x,y,\chi,\theta} J(N,\overline{\chi})J(\overline{N},\overline{\theta})\overline{L}N\overline{\chi}\theta(x)\overline{M}\chi\theta(y).$$

The x and y sums are 0 unless $\overline{L}N\overline{\chi}\theta = \varepsilon$ and $\overline{M}\chi\theta = \varepsilon$, so we must have $\theta^2 = LM\overline{N}$ and $\chi = M\overline{\theta}$. Note $\theta^2 = (\varphi\theta)^2$. We have

$$\text{(4.18)}\quad \sum_{x,y} \overline{L}(x)M(1-x)\overline{M}(y)L(1-y)N(x-y)$$
$$= (q-1)\delta(N) - (q-1)\delta(M) - (q-1)N(-1)\delta(L)$$
$$+ \begin{cases} 0, & \text{if } LM\overline{N} \text{ is not a square,} \\ J(N,\overline{M}R)J(\overline{N},\overline{R}) + J(N,\varphi\overline{M}R)J(\overline{N},\varphi\overline{R}), & \text{if } LM\overline{N} = R^2. \end{cases}$$

Similar results to (4.18) can be found in [**2**, (5)] and [**8**, 4.37]. If we use (4.1) in the left hand side of (4.18), appeal to properties (1.10) and (1.11) for Jacobi sums and convert Jacobi sums to binomial coefficients, we have

$$\text{(4.19)}\quad \frac{q}{q-1}\sum_{\chi}\binom{N}{\chi}\binom{M}{L\overline{N}\chi}\binom{L}{L\overline{M}\chi}\chi(-1)$$
$$= \frac{q-1}{q^2}LM(-1)\delta(N) - \frac{q-1}{q^2}LN(-1)\delta(M)$$
$$- \frac{q-1}{q^2}M(-1)\delta(L)$$
$$+ \begin{cases} 0, & \text{if } LM\overline{N} \text{ is not a square,} \\ MR(-1)\binom{NR}{N}\binom{N}{L\overline{R}} + \varphi MR(-1)\binom{\varphi NR}{N}\binom{N}{\varphi L\overline{R}}, & \text{if } LM\overline{N} = R^2. \end{cases}$$

This is an analogue for the binomial coefficient version of Dixon's theorem [**6**, (2)]

$$(4.20)\quad \sum_{k\geq 0}\binom{n}{k}\binom{m}{l-n+k}\binom{l}{l-m+k}(-1)^k$$
$$=\begin{cases}0, & \text{if } l+m-n \text{ is odd,}\\ (-1)^{m-r}\binom{n+r}{n}\binom{n}{l-r}, & \text{if } l+m-n=2r.\end{cases}$$

Finally, if we take

$$f(x,y)=\varepsilon(1-x)\varepsilon(1-y)\overline{L}(x)\overline{M}(y)\overline{N}(1-xy)$$
$$\times N(x-\alpha y-(1-\alpha)xy),$$

then

$$f\left(\frac{x}{1-y},\frac{y}{1-x}\right)$$
$$=\varepsilon(1-x-y)\overline{L}(x)M(1-x)\overline{M}(y)L(1-y)N(x-\alpha y).$$

If we take $\alpha\neq 0$, then with calculations similar to the previous example,

$$\text{C.T.}\, f\left(\frac{x}{1-y},\frac{y}{1-x}\right)$$
$$=\frac{L(-1)}{(q-1)^3}\sum_{\chi}J(N,\overline{\chi})J(M,\overline{L}N\overline{\chi})J(L,\overline{L}M\overline{\chi})\chi(\alpha)$$
$$-\frac{q}{(q-1)^2}\delta(1+\alpha)-\frac{1}{q-1}\varepsilon(1+\alpha)\delta(N)$$
$$+\frac{1}{(q-1)^2}N(-\alpha)+\frac{1}{(q-1)^2},$$

and

$$\text{C.T.}\, f(x,y)=\sum_{\chi}J(L\overline{M}\overline{N}\chi^2,\overline{\chi})J(N,\overline{L}M\overline{\chi})J(\overline{N},\overline{L}N\overline{\chi})\chi\left(\frac{\alpha}{(1-\alpha)^2}\right)$$
$$+c_2\begin{cases}0, & \text{if } \overline{L}MN \text{ is not a square}\\ J(N,\overline{R})J(\overline{N},\overline{M}R)+J(N,\varphi\overline{R})J(\overline{N},\varphi\overline{M}R), & \\ & \text{if } \overline{L}MN=R^2\end{cases}$$
$$-\frac{1}{q-1}\delta(M)-\frac{N(-\alpha)}{q-1}\delta(L)+\frac{1}{(q-1)^2}+\frac{N(-\alpha)}{(q-1)^2}$$
$$+\frac{1}{(q-1)^2}\overline{L}MN\left(\frac{-\alpha}{1-\alpha}\right)J(\overline{M},\overline{N})$$

where

$$c_1 = \frac{N(-1)}{(q-1)^3}\overline{L}MN(1-\alpha),$$

and

$$c_2 = \frac{L(-1)}{(q-1)^2}\delta(1-\alpha).$$

By Theorem 4.9, after converting Jacobi sums to binomial coefficients,

$$\begin{aligned}(4.21)\quad \frac{q}{q-1}\sum_{\chi}\binom{N}{\chi}\binom{M}{L\overline{N}\chi}\binom{L}{L\overline{M}\chi}\chi(-\alpha) &= E_1 + E_2 + LN(-1)\overline{L}MN(1-\alpha)\frac{q}{q-1} \\ &\times\sum_{\chi}\binom{L\overline{MN}\chi^2}{\chi}\binom{N}{L\overline{M}\chi}\binom{\overline{N}}{L\overline{N}\chi}\chi\left(\frac{\alpha}{(1-\alpha)^2}\right)\end{aligned}$$

where E_1 and E_2 are the "error" terms

$$\begin{aligned}E_1 = {}& \frac{q-1}{q^2}LM(-1)\varepsilon(1+\alpha)\delta(N) \\ &- \frac{q-1}{q^2}LN(-1)\delta(M) - \frac{q-1}{q^2}M(-1)N(\alpha)\delta(L) \\ &+ \frac{LMN(-1)}{q}\delta(1+\alpha) + \frac{1}{q^2}\overline{L}MN\left(\frac{\alpha}{1-\alpha}\right)J(\overline{M},\overline{N})\end{aligned}$$

and

$$E_2 = \delta(1-\alpha)\begin{cases}0, & \text{if } LM\overline{N} \text{ is not a square} \\ MR(-1)\binom{NR}{N}\binom{N}{L\overline{R}} + \varphi MR(-1)\binom{\varphi NR}{N}\binom{N}{\varphi L\overline{R}}, & \text{if } LM\overline{N} = R^2.\end{cases}$$

When $\alpha = 1$, this reduces to (4.15). For general α, (4.17) is an analogue for the following version of Whipple's ${}_3F_2$-quadratic transformation [**6**, (11)]

$$\begin{aligned}(4.22)\quad &\sum_{k\geq 0}\binom{n}{k}\binom{l}{l-m+k}\binom{l}{l-m-k}(-\alpha)^k \\ &= (1-\alpha)^{m+n-l} \\ &\times \sum_{j\leq(m+n-l)/2}\frac{(l+j)!}{j!(l-n+j)!(l-n+j)!(m+n-l-2j)!}\left(\frac{-\alpha}{(1-\alpha)^2}\right)^j.\end{aligned}$$

The identities (4.14), (4.19) and (4.21) indicate that there is an analogue for hypergeometric series over finite fields. In fact, **[8]** describes such an analogue in which these results are proved, by different methods, as hypergeometric series identities.

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