Polynomial Factorizations and Character Sum Identities

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1 INTRODUCTION AND NOTATION

It is well known that hypergeometric series identities can be derived from polynomial identities. For example, [8, p.238 problem 29]: When both sides of

$$(1.1) \qquad (1-x)^n(1+x)^n = (1-x^2)^n$$

are expanded by the binomial theorem and coefficients of x^r are equated, we have

$$(1.2) \qquad \sum_{k=0}^{r} \binom{n}{k}\binom{n}{r-k}(-1)^k = \begin{cases} 0, & \text{if } r \text{ is odd} \\ \binom{n}{r/2}(-1)^{r/2}, & \text{if } r \text{ is even.} \end{cases}$$

This expression can be recast in terms of hypergeometric series as

$$(1.3) \qquad {}_2F_1\left(\begin{matrix} -n, & -r \\ & n-r+1 \end{matrix}\,\middle|\, -1\right) = \begin{cases} 0, & \text{if } r \text{ is odd,} \\ \dfrac{r!\,(n-r)!}{(r/2)!\,(n-r/2)!}, & \text{if } r \text{ is even.} \end{cases}$$

Integral evaluations may also be obtained in this way. For example,

$$\int_0^1 x^b(1-x)^a(1+x)^a \frac{dx}{x(1-x)} = \int_0^1 x^b(1-x^2)^a \frac{dx}{x(1-x)}, \tag{1.4}$$

if $a, b > 0$. Substituting $u = x^2$, the right hand side of (1.4) becomes

$$\int_0^1 u^{b/2}(1-u)^a \frac{du}{2u(1-\sqrt{u})}$$

$$= \int_0^1 u^{b/2}(1-u)^a \frac{du}{2u(1-u)}(1+\sqrt{u})$$

$$= \frac{\beta(b/2, a) + \beta(1/2 + b/2, a)}{2},$$

where β is the beta function, $\beta(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$. Hence

$$\int_0^1 x^b(1-x)^a(1+x)^a \frac{dx}{x(1-x)} = \frac{\beta(b/2, a) + \beta(1/2 + b/2, a)}{2}. \tag{1.5}$$

In this paper, we proceed analogously to produce identities for character sums over finite fields. Specifically, we are looking primarily for evaluations of expressions of the type

$$\sum_x A(x)B(1-x)C(f(x)),$$

where A, B and C are multiplicative characters of GF(q). It is surprising that interesting evaluations can arise by way of factoring polynomials. Nevertheless, many formulas derived in this paper are new and are difficult to derive by other means. The most interesting results contained in this paper are (3 .12), (3 .13), (3 .19) and (3 .20). Formulas (2.8), (2.12), (2.13), (3 .7) and (3 .11) also appear to be new. This paper is organised as follows. In section 2, two infinite families of identities are used and specialized. In section 3, more exotic identities are employed, and in section 4, we present analogies with [illegible]al evaluations. In the remainder of this section, we define the notation to be [illegible] this paper.

Let GF(q) denote the finite field with q elements, where q is a power of an odd prime p. Let Tr be the trace function from GF(q) to GF(p). Throughout this paper, capital letters A, B, C, $\cdots$ and the Greek letters χ, ψ, ϕ and ε will denote multiplicative characters of GF(q). We will let δ denote both the function on GF(q) defined by

$$\delta(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and the function on multiplicative characters defined by

$$\delta(A) = \begin{cases} 1, & \text{if } A \text{ is the trivial character,} \\ 0, & \text{otherwise.} \end{cases}$$

The trivial character will be denoted ε, the quadratic character ϕ and we will let ψ denote a character of order dividing $q - 1$. Define $\overline{A}$ by $A\overline{A} = \varepsilon$, and extend the definition of all multiplicative characters to 0 by $A(0) = 0$.

Write $\sum_x$ to denote the sum over all x in GF(q) and $\sum_\chi$ to denote the sum over all multiplicative characters of GF(q). Let $\zeta = e^{2\pi i/p}$. The Gauss sum, G(A) is defined by

$$G(A) = \sum_x A(x)\,\zeta^{\mathrm{Tr}(x)} \tag{1.6}$$

and the Jacobi sum, J(A, B) is defined by

$$J(A, B) = \sum_x A(x)B(1 - x). \tag{1.7}$$

Informally, A(x) is analogous to x^n, G(A) is analogous to $\Gamma(a)$ and J(A, B) is analogous to $\beta(a, b)$. See [1], [2], [5] for examples of this analogy. We will make use of the following properties of Gauss and Jacobi sums [ections 5.2, 5.3]:

$$G(A)G(\overline{A}) = qA(-1) - (q - 1)\delta(A) \tag{1.8}$$

$$J(A,B) = J(B,A) = J(\overline{AB}, B)\,B(-1), \tag{1.9}$$

$$J(A,B) = \frac{G(A)G(B)}{G(AB)} + (q-1)B(-1)\delta(AB), \tag{1.10}$$

and

$$J(A,\overline{A}) = -A(-1) + (q-1)\delta(A). \tag{1.11}$$

We will also make use of the orthogonality relations [7, page 188]

$$\sum_{x} A(x) = (q-1)\,\delta(A), \tag{1.12}$$

$$\sum_{\chi} \chi(x) = (q-1)\,\delta(1-x), \tag{1.13}$$

and the formula

$$\sum_{x} f(x^n) = \sum_{x} f(x)(1 + \psi(x) + \psi^2(x) + \cdots + \psi^{d-1}(x)), \tag{1.14}$$

where $d = \gcd(n, q-1)$ and ψ is a character of order d.

2 INFINITE FAMILIES OF FACTORIZATIONS

There are two infinite families of factorizations suitable for obtaining character sum identities. They are

$$(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1-x^n \tag{2.1}$$

and

$$(1+x)^n - x^n = \sum_{k=0}^{n-1} \binom{n}{k} x^k. \tag{2.2}$$

If we apply A to both sides of (2.1), multiply by B(x) and sum we have

$$\sum_x B(x)A(1 - x)A(1 + x + x^2 + \cdots + x^{n-1}) = \sum_x B(x)A(1 - x^n).$$

Let $d = \gcd(n, q - 1)$. If B is not a d'th power, this sum is 0, for if r is a primitive d'th root of unity in GF(q) and we make the change of variables $x \to rx$, we have

$$\sum_x B(x)A(1 - x^n) = B(r)\sum_x B(x)A(1 - x^n).$$

Thus,

$$(1 - B(r))\sum_x B(x)A(1 - x^n) = 0.$$

Since $B(r) = 1$ if and only if B is a d'th power, the claim is established. If B is a d'th power, it is also an n'th power so $B = C^n$ for some character C. If ψ has order d then by (1.14),

$$\begin{aligned}\sum_x B(x)A(1 - x^n) &= \sum_x C(x^n)A(1 - x^n)\\ &= \sum_x C(x)A(1 - x)(1 + \psi(x) + \cdots + \psi^{d-1}(x)).\end{aligned}$$

Hence,

$$\sum_x B(x)A(1 - x)A(1 + x + x^2 + \cdots + x^{n-1}) = \tag{2.3}$$

$$\begin{cases} 0, & \text{if } B \text{ is not an n'th power,}\\ J(A, C) + J(A, \psi C) + \cdots + J(A, \psi^{d-1}C), & \text{if } B = C^n, \end{cases}$$

where $d = \gcd(n, q - 1)$ and ψ has order d.

Formula (2.3) has the following special cases. When $n = 2$, we have

$$(2.4)\quad \sum_x B(x)A(1-x)A(1+x) = \begin{cases} 0, & \text{if } B \text{ is not a square,} \\ J(A, C) + J(A, \phi C), & \text{if } B = C^2. \end{cases}$$

Changing x to $\frac{-x}{1-x}$ gives

$$(2.5)\quad \sum_x B(x)\overline{A}^2\overline{B}(1-x)A(1-2x) = \begin{cases} 0, & \text{if } B \text{ is not a square,} \\ J(A, C) + J(A, \phi C), & \text{if } B = C^2. \end{cases}$$

If $B = \overline{A}^2$, we obtain the well-known formula [7, exercise 5.47]

$$(2.6)\quad J(\overline{A}^2, A) = (q-1)\delta(A) + \overline{A}(4)J(A, \phi\overline{A}).$$

When $n = 3$,

$$(2.7)\quad \sum_x B(x)A(1-x)A(1+x+x^2) = H(A, B),$$

where

$$H(A, B) = \begin{cases} J(A, C), & \text{if } 3 \text{ does not divide } q-1 \text{ and } B = C^3, \\ 0, & \text{if } 3 \text{ divides } q-1 \text{ and } B \text{ is not a cube,} \\ J(A, C) + J(A, \psi C) + J(A, \psi^2 C), & \text{if } 3 \text{ divides } q-1 \text{ and } B = C^3. \end{cases}$$

After replacing x by $(x-1)/2$, we get,

$$(2.8)\quad \sum_x B(1-x)A(3-x)A(3+x^2) = A(8)\,B(-2)\,H(A, B).$$

From (2.2) we have

$$(2.9)\quad \sum_x B(x)A((1+x)^n - x^n) = \sum_x B(x)A\Big(\sum_{k=0}^{n-1}\binom{n}{k}x^k\Big).$$

The left hand side of (2.9) can be written

$$\sum_x B(x)A((1+x)^n - x^n)$$

$$= B(-1)A^{n+1}(-1) + \sum_x B(x)A^n(1+x)A(1 - \frac{x^n}{(1+x)^n})$$

$$= B(-1)A^{n+1}(-1) + \sum_x B(x)\overline{A}^{\,n}\overline{B}(1-x)A(1-x^n).$$

When $n = 2$, the right hand side of (2.9) becomes

$$\sum_x B(x)A(1+2x) = \overline{B}(-2)J(A, B)$$

so

$$\sum_x B(x)\overline{AB}(1-x)A(1+x) = -AB(-1) + \overline{B}(-2)J(A, B). \tag{2.10}$$

When $n = 3$, we have

$$\sum_x B(x)\overline{A}^{\,3}\overline{B}(1-x)A(1-x^3) + B(-1) = \sum_x B(x)A(1+3x+3x^2).$$

Changing x to $\frac{x-1}{2}$ on the right gives

$$\sum_x B(1-x)A(1+3x^2) = A(4)B(2) + A(4)B(-2)\sum_x B(x)\overline{A}^{\,3}\overline{B}(1-x)A(1-x^3). \tag{2.11}$$

The right hand side of (2.11) is evaluable if $B = \overline{A}^{\,2}$ or $B = \overline{A}^{\,3}$. In the first case, we have

$$\sum_x \overline{A}^{\,2}(1-x)A(1+3x^2) = -A(3) + (q-1)\delta(A) + A(\tfrac{3}{4})\phi(-3)J(A, \phi). \tag{2.12}$$

In the second case,

$$(2.13)\qquad \sum_x \overline{A}^{\,3}(1-x)A(1+3x^2) =$$

$$(q-1)\delta(A) + \overline{A}(2)\begin{cases} 0, & \text{if } 3 \text{ does not divide } q-1, \\ J(A,\psi) + J(A,\psi^2), & \text{if } 3 \text{ divides } q-1, \end{cases}$$

where ψ has order 3. In general, if $B = \overline{A}^{\,n}$ and $d = \gcd(n, q-1)$, we have

$$(2.14)\qquad \sum_x \overline{A}^{\,n}(x)A((1+x)^n - x^n)$$

$$= (q-1)\delta(A) + A(-1)(J(A,\psi) + J(A,\psi^2) + \cdots + J(A,\psi^{d-1})),$$

where ψ has order d.

3 SPECIAL FACTORIZATIONS

The evaluations in this section are based on factorizations of expressions of the form

$$(1-x)^n + ax^m(1-x)^k$$

for special choices of n, a, m and k. Some trivial examples are

$$(3.1)\qquad (1-x)^2 + 2x = 1 + x^2,$$

$$(3.2)\qquad (1-x)^2 + 2x(1-x) = 1 - x^2$$

and

$$(3.3)\qquad (1-x)^2 + 4x = (1+x)^2.$$

More substantial are

$$(1 - x)^3 + 3x(1 - x) = 1 - x^3, \tag{3.4}$$

$$(1 - x)^3 + \frac{27}{4}x = (1 + 2x)^2\left(1 - \frac{x}{4}\right) \tag{3.5}$$

and

$$(1 - x)^4 - 16x^2 = (1 + x)^2(1 - 6x + x^2). \tag{3.6}$$

With (3.1) we have

$$\sum_x B(x)A((1 - x)^2 + 2x) = \sum_x B(x)A(1 + x^2).$$

The right hand side of this expression is 0 if B is not a square and can be evaluated by (1.14) if B is a square. On the left hand side, we have

$$\sum_x B(x)A((1 - x)^2 + 2x) = A(2) + \sum_x B(x)A^2(1 - x)A\left(1 + \frac{2x}{(1 - x)^2}\right).$$

Changing x to $\frac{-1 + x}{1 + x}$, we have

$$\sum_x B(1 - x)\overline{A}^2\overline{B}(1 + x)A(1 + x^2) = \tag{3.7}$$

$$- B(-1) + \overline{A}(2)\begin{cases} 0, & \text{if } B \text{ is not a square,} \\ C(-1)J(A, C) + \varphi C(-1)J(A, \varphi C), & \text{if } B = C^2. \end{cases}$$

If $B = \overline{A}$, the left hand side of (3.7) becomes

$$\sum_x \overline{A}(1 - x^2)A(1 + x^2)$$

which partially evaluates via (1.14) to give

$$(3.8)\qquad \sum_x \phi(x)\overline{A}(1-x)A(1+x) =$$

$$-(q-1)\delta(A) + \overline{A}(2)\begin{cases} 0, & \text{if } A \text{ is not a square,} \\ C(-1)J(A,\overline{C}) + \varphi C(-1)J(A,\varphi\overline{C}), & \text{if } A = C^2. \end{cases}$$

Factorization (3.2) leads to (2.5); with (3.3) we have

$$\sum_x B(x)A((1-x)^2+4x) = \sum_x B(x)A^2(1+x) = B(-1)J(A^2, B).$$

The usual methods now give

$$(3.9)\qquad \sum_x A^2(x)B(1-x)\overline{A}^2\overline{B}(1+x) = -B(-1) + \overline{A}(4)J(A^2, B).$$

With (3.4), the evaluations become more interesting. We have

$$\sum_x B(x)A((1-x)^3 + 3x(1-x)) = \sum_x B(x)A(1-x^3) = H(A, B),$$

with H as in (2.7). Changing x to $-\frac{1-x}{1+x}$ gives

$$(3.10)\qquad \sum_x B(1-x)\overline{A}^3\overline{B}(1+x)A(1+3x^2) = B(-1)\overline{A}(2)\,H(A, B).$$

There are two interesting special cases for (3.10). First, if $B = \overline{A}^3$, then

$$(3.11)\qquad \sum_x \overline{A}^3(1-x)A(1+3x^2) =$$

$$(q-1)\delta(A) + \begin{cases} 0, & \text{if } 3 \text{ does not divide } q-1, \\ J(A,\psi\overline{A}) + J(A,\psi^2\overline{A}), & \text{if } 3 \text{ divides } q-1, \end{cases}$$

where ψ has order 3. The second case is where $B = \overline{A}^3\overline{B}$. If we replace A by A^2 and B by $\overline{A}^3$ in (3.10), then we have

$$\sum_x \overline{A}^3(1-x^2)A^2(1+3x^2) = \overline{A}(-4)\,H(A^2,\overline{A}^3).$$

The sum can be partially evaluated by (1.14),

$$\sum_x \overline{A}^3(1-x^2)A^2(1+3x^2)$$

$$= \sum_x \overline{A}^3(1-x)A^2(1+3x)(1+\phi(x))$$

$$= A(\tfrac{27}{4})J(A^2,\overline{A}^3) + \sum_x \phi(x)\overline{A}^3(1-x)A^2(1+3x).$$

Thus,

$$(3.12)\qquad \sum_x \phi(x)\overline{A}^3(1-x)A^2(1+3x) = -A(\tfrac{27}{4})J(A^2,\overline{A}^3) + \overline{A}(-4)\,H(A^2,\overline{A}^3)$$

or

$$(3.13)\qquad \sum_x \overline{A}^3(x)A^2(1-x)\phi(1-\tfrac{4}{3}x) = -J(A^2,\overline{A}^3) + \overline{A}(-27)\,H(A^2,\overline{A}^3).$$

A more complicated derivation is required for (3.5). We have

$$\sum_x B(x)A^2(1+2x)A(1-\tfrac{x}{4}) = \sum_x B(x)A((1-x)^3+\tfrac{27}{4}x)$$

$$= A(\tfrac{27}{4}) + \sum_x B(x)A^3(1-x)A(1+\frac{27x}{4(1-x)^3}).$$

Changing x to 4x on the left hand side,

$$(3.14)\qquad \sum_x B(x)A(1-x)A^2(1+8x)$$

$$= \overline{A}^3(4)A(27) + \overline{B}(4)\sum_x B(x)A^3(1-x)A(1+\frac{27x}{4(1-x)^3}).$$

In [4, Theorem 2.3], the following analog for the binomial theorem for multiplicative characters is given:

$$(3.15) \qquad A(1+x) = \delta(x) + \frac{1}{q-1}\sum_{\chi} J(A, \bar{\chi})\, \chi(-x).$$

Applying this to the sum on the right hand side of (3 .14) gives

$$\sum_{x} B(x)A^3(1-x)A\left(1 + \frac{27x}{4(1-x)^3}\right)$$

$$= \frac{1}{q-1}\sum_{x,\,\chi} J(A, \bar{\chi})B(x)A^3(1-x)\chi\left(-\frac{27x}{4(1-x)^3}\right)$$

$$= \frac{1}{q-1}\sum_{\chi} J(A, \bar{\chi})\, J(B\chi, A^3\bar{\chi}^3)\, \chi\left(-\frac{27}{4}\right).$$

With (1.9) and (1.10),

$$\frac{1}{q-1}\sum_{\chi} J(A, \bar{\chi})\, J(B\chi, A^3\bar{\chi}^3)\, \chi\left(-\frac{27}{4}\right)$$

$$= (q-1)A\left(\frac{27}{4}\right)\delta(AB) - A\left(\frac{27}{4}\right) + B\left(-\frac{4}{27}\right)\frac{G(A)G(B)}{G(AB)}$$

$$+ A(-1)G(A)\frac{1}{q-1}\sum_{\chi}\frac{G(\bar{A}^3\bar{B}\chi^2)G(A^3\bar{\chi}^3)G(\bar{\chi})}{G(A\bar{\chi})G(\bar{B}\chi)}\,\chi\left(\frac{27}{4}\right).$$

The sum above can be simplified using the replication formulas [7, Theorem 5.28]

$$(3.16) \qquad G(A^2) = \frac{G(A)G(\phi A)}{G(\phi)}A(4)$$

and

$$(3.17) \qquad G(A^3) = \frac{G(A)G(\psi A)G(\psi^2 A)}{G(\psi)G(\psi^2)}A(27),$$

if we assume $B = \psi\bar{A}$. We have

$$\frac{1}{q-1}\sum_{\chi}\frac{G(\psi^2\bar{A}^2\chi^2)G(A^3\bar{\chi}^3)G(\bar{\chi})}{G(A\bar{\chi})G(\psi^2A\bar{\chi})}\chi(\frac{27}{4})$$

$$= \psi(4)A(\frac{27}{4})\frac{1}{q-1}\sum_{\chi}\frac{G(\psi\bar{A}\chi)G(\varphi\psi\bar{A}\chi)G(\psi A\bar{\chi})G(\bar{\chi})}{G(\varphi)G(\psi)G(\psi^2)}.$$

Finally, this last sum may be evaluated by the formula [6, Theorem 2]

$$(3.18)\qquad \frac{1}{q-1}\sum_{\chi}G(A\chi)G(B\chi)G(C\bar{\chi})G(D\bar{\chi})$$

$$= q(q-1)CD(-1)\delta(ABCD) + \frac{G(AC)G(BC)G(AD)G(BD)}{G(ABCD)}$$

to obtain

$$\frac{1}{q-1}\sum_{\chi}\frac{G(\psi^2\bar{A}^2\chi^2)G(A^3\bar{\chi}^3)G(\bar{\chi})}{G(A\bar{\chi})G(\psi^2A\bar{\chi})}\chi(\frac{27}{4})$$

$$= (q-1)\frac{\psi(4)\varphi(-3)}{G(\varphi)}\delta(\phi\bar{A}) + \psi(4)A(\frac{27}{4})\frac{G(\psi\bar{A})G(\varphi\psi\bar{A})G(\psi^2)G(\varphi\psi^2)}{G(\varphi)G(\psi)G(\psi^2)G(\varphi\bar{A})}$$

$$= (q-1)\frac{\psi(4)\varphi(-3)}{G(\varphi)}\delta(\phi\bar{A}) + \psi(4)A(27)\frac{G(\psi^2\bar{A}^2)G(\varphi)}{G(\psi^2)G(\varphi\bar{A})}.$$

Thus,

$$\sum_{x}\psi\bar{A}(x)A(1-x)A^2(1+8x) =$$

$$(q-1)\phi(-1)\delta(\phi A) + A(-27)\frac{G(A)G(\psi\bar{A})}{G(\psi)} + A(27)A(-4)\frac{G(\psi^2\bar{A}^2)G(\varphi)G(A)}{G(\psi^2)G(\varphi\bar{A})},$$

which simplifies to

$$(3.19)\qquad \sum_{x}\psi\bar{A}(x)A(1-x)A^2(1+8x) = A(-27)\frac{G(A)G(\psi\bar{A})}{G(\psi)} + A(27)\frac{G(A^2)G(\psi^2\bar{A}^2)}{G(\psi^2)}.$$

Along the same lines, with (3 .6) we have

$$\sum_x B(x)A^2(1+x)A(1-6x+x^2) = \sum_x B(x)A((1-x)^4 - 16x^2)$$

$$= A(-16) + \sum_x B(x)A^4(1-x)A\left(1 - \frac{16x^2}{(1-x)^4}\right)$$

$$= A(-16) + \frac{1}{q-1}\sum_\chi J(A,\bar{\chi})\, J(B\chi^2, A^4\bar{\chi}^4)\,\chi(16).$$

Let $B = \phi\bar{A}^2$ and expand this sum using (1.10). Assume for simplicity, that 4 divides $q - 1$, and let $\phi = \alpha^2$. Then several applications of (3 .16) yields

$$\sum_x \phi\bar{A}^2(x)A^2(1+x)A(1-6x+x^2) = A(16)\frac{G(A)G(\alpha\bar{A})}{G(\alpha)} + A(16)\frac{G(A)G(\alpha\varphi\bar{A})}{G(\alpha\varphi)}$$

$$+ \frac{G(A)}{G(\varphi)}\phi(2)A(16)\frac{1}{q(q-1)}\sum_\chi G(\alpha\bar{A}\chi)G(\alpha\phi\bar{A}\chi)G(\phi A\bar{\chi})G(\bar{\chi}).$$

Using (3 .18) and simplifying gives

$$(3\ .20)\qquad \sum_x \phi\bar{A}^2(x)A^2(1-x)A(1+6x+x^2)$$

$$= A(16)\frac{G(A)G(\alpha\bar{A})}{G(\alpha)} + A(16)\frac{G(A)G(\alpha\varphi\bar{A})}{G(\alpha\varphi)} + \phi(2)A(-1)\frac{G(A^2)G(\varphi\bar{A}^2)}{G(\varphi)}.$$

4 COMMENTS

There are hypergeometric series identities and integral evaluations analogous to the evaluations given in this paper. We give some of them here.

Analogous to (2.3) is the integral formula

$$(4.1)\qquad \int_0^1 x^b(1-x)^a(1+x+\cdots+x^{n-1})^a \frac{dx}{x(1-x)}$$

$$= \frac{1}{n}\left(\beta(a, \tfrac{b}{n}) + \beta(a, \tfrac{1+b}{n}) + \cdots + \beta(a, \tfrac{n-1+b}{n})\right),$$

provided $a, b > 0$. From (4.1) it follows that for $a, b > 0$,

$$(4.2)\qquad \int_0^\infty x^b(1+x)^{-2a-b}(1+2x)^a \frac{dx}{x} = \frac{\beta(b/2, a) + \beta(1/2 + b/2, a)}{2}$$

and

$$(4.3)\qquad \int_1^3 (x-1)^b(3-x)^a(3+x^2)^a \frac{dx}{(x-1)(3-x)}$$

$$= \frac{1}{3}\, 2^{3a+b-1}\left(\beta(a, \tfrac{b}{3}) + \beta(a, \tfrac{1+b}{3}) + \beta(a, \tfrac{2+b}{3})\right).$$

These are analogous to (2.5) and (2.8), respectively.

Formula (2.9) and its special cases (2.12) and (2.13) do not appear to have natural integral analogs. However, analogous to (2.10) is

$$(4.4)\qquad \int_0^1 x^b(1-x)^{-a-b}(1+x)^a \frac{dx}{x(1-x)} = 2^{-b}\beta(b, -a-b),$$

which holds for $b > 0$, $a + b < 0$. Analogous to (3.10) is

$$(4.5)\qquad \int_1^\infty (x-1)^b(x+1)^{-b-3a}(3x^2+1)^a \frac{dx}{x-1} =$$

$$\frac{1}{3}\, 2^{-a}\left(\beta(a, \tfrac{b}{3}) + \beta(a, \tfrac{1+b}{3}) + \beta(a, \tfrac{2+b}{3})\right),$$

provided $a, b > 0$. Unfortunately, the specializations that gave rise to (3.11) through (3.13) do not work for integrals. This is because the substitutions

$B = \overline{A}^{\,k}$ in sections 3 correspond to $b = -ka$ in (4.5), violating the conditions needed for the integral to converge. Also, if we attempt to use (3 .5) in an integral evaluation, there are problems. Specifically, we have

$$\int_0^1 x^b(1-x)^a(1+8x)^{2a}\frac{dx}{x(1-x)} = \int_0^1 x^b((1-4x)^3+27x)^a\frac{dx}{x(1-x)},$$

but $1-4x$ has a zero on the interval of integration so the algebraic manipulations involved in obtaining (3 .19) are invalid for the analogous integrals. However, these formulas do have hypergeometric series evaluation analogs. Among them are [3, eq. (1.1)]

$$(4.6) \qquad {}_3F_2\left(\begin{matrix}-n, & n+3a, & a \\ & 3a/2, & (3a+1)/2\end{matrix}\ \middle|\ \frac{3}{4}\right)$$

$$= \begin{cases} 0, & \text{if } 3 \text{ does not divide } n, \\ \dfrac{n!(a+1)_m}{m!(3a+1)_n}, & \text{if } n = 3m, \end{cases}$$

and [3, eq. (3.7)]

$$(4.7) \qquad {}_2F_1\left(\begin{matrix}-n, & -2n-2/3 \\ & 4/3\end{matrix}\ \middle|\ -8\right) = (-27)^n\frac{(\frac{5}{6})_n}{(\frac{3}{2})_n}.$$

In the above, $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$. Formula (4.6) is the hypergeometric series analog of (3 .10). Formula (3 .11) is analogous to the specialization $a = -n/3$ in (4.6), formulas (3 .12) and (3 .13) are analogous to the specialization $a = -2n/3$. Formula (4.7) is the analog of (3 .19).

The author could not find a way to evaluate the right hand side of (3 .14) without resorting to formulas (3 .15) through (3 .18). It would be very interesting to evaluate this sum by other methods.

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