4.5 8 measurements of the period T of a pendulum

$N = 8$

$$\{T_i\} = \{1.35, 1.34, 1.32, 1.36, 1.33, 1.34, 1.37, 1.35\} \text{ seconds}$$

(a)

$$\mu_T \approx \bar{T} = \frac{1}{N}\sum_{i=1}^{N} T_i = \frac{1}{8}\sum_{i=1}^{N} T_i = 1.345\ s \quad \text{via calculator}$$

$$\sigma_T^2 \approx S_T^2 = \frac{1}{N-1}\sum_{i=1}^{N}(T_i - \bar{T})^2 \qquad \sigma_T \approx S_T = 0.016\ s \quad \text{via calculator}$$

Std deviation of the mean $\sigma_\mu = \frac{\sigma_T}{\sqrt{N}} \approx \frac{S_T}{\sqrt{8}}$

$$\sigma_\mu = \frac{0.016\ s}{\sqrt{8}} = 0.0057\ s$$
$$(\approx 0.006\ s)$$

(b) Prob of next measurement falling with .02 s of mean

$$= \int_{\bar{T}-.02s}^{\bar{T}+.02} P(T)\, dT \approx P(\bar{T})\, \Delta T$$

$$\Delta T \approx .04 s$$

but $P(T)$ here should be Gaussian

$$P(T) = \frac{1}{\sqrt{2\pi\sigma_T^2}} e^{-\frac{1}{2}\frac{(T-\bar{T})^2}{\sigma_T^2}}$$

Use dimensionless variable $z \equiv \left(\frac{T-\bar{T}}{\sigma_T}\right)^2$

$$T - \bar{T} = .02 \Rightarrow z = \frac{.020}{.016} = 1.25$$

what's the prob. of next measurement being within $\pm 1.25\sigma$ of mean?

$$P = \frac{1}{\sqrt{2\pi}} \int_{-1.25}^{1.25} e^{-\frac{1}{2}z^2} \, dz = 0.789 \approx 79\%$$

chance

(Consult table C.2 with $z = 1.25$)

(Recall $\approx 2/3$ (68%) of measurements expected within $\pm 1\sigma$ if Gaussian distribution applies. expect answer > 68%)

4.7 Counts of decays in 1 min interval

$$\{x_i\} = \{125, 130, 105, 126, 128, 119, 137, 131, 115, 116\}$$

(counts per minute)

Background measured separately was 58 counts in 5 minutes

$b = 58$ Recorded in 5 minutes

(a) b per minute: $b' = \frac{58}{5} = 11.6$ (counts/minute)

Flutuations in per minute background

$\sigma_b = \sqrt{b}$ for Poisson distribution relevant to this situation

$\sigma_b = \sqrt{58} = 7.6$

Fluctuations in a 5 minute count would b ± 7.6

But for a 1 minute interval we want

$\sigma_{b'} = \sqrt{b'} = \sqrt{11.6} \frac{\text{counts}}{\text{minute}} = \sigma_{b'} = 3.41$ counts in one minute

$b' = 11.6 \pm 3.48 \frac{\text{counts}}{\text{minute}}$

$\bar{x}_0 = \frac{1}{N}\sum_{i=1}^{N} x_i$ $N = 10$ measurements (above)

Via calculator

$\bar{x} = 123.2 \frac{\text{counts}}{\text{minute}}$ $\sigma_x \simeq s_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$

$\sigma_x \simeq s_x = 9.4$

Corrected rate $R = x - b' = \bar{x} - b' = 123.2 - 11.6 = 111.6$

$$\sigma_R^2 = \left(\frac{\partial R}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial R}{\partial b'}\right)^2 \sigma_{b'}^2 \qquad \frac{\partial R}{\partial x} = 1 \quad \frac{\partial R}{\partial b'} = -1$$

$$\sigma_R = \sqrt{\sigma_x^2 + \sigma_{b'}^2} = \sqrt{(9.4)^2 + (3.4)^2} = 10.0$$

So $\bar{R} = \bar{x} - b' = (123.2 - 11.6) = 111.6 \pm 10.0$ counts in 1 minute

6.4 $y = bx \qquad \{x_i, y_i\}$

$$\chi^2 = \sum_{i=1}^{N} \frac{[y_i - y(x_i)]^2}{\sigma_i^2}$$

$$= \sum_{i=1}^{N} \frac{(y_i - bx_i)^2}{\sigma_i^2}$$

Minimize χ^2 w.r.t b to maximize probability of the given data set $\{x_i, y_i\}$

$$\frac{\partial \chi^2}{\partial b} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} 2(y_i - bx_i)(-x_i) = 0$$

$$\Rightarrow \quad -\sum_{i=1}^{N} \frac{2 y_i x_i}{\sigma_i^2} + b \sum_{i=1}^{N} \frac{2}{\sigma_i^2} x_i^2 = 0$$

$\Rightarrow$

$$b = \frac{+\sum_i^N x_i y_i / \sigma_i^2}{\sum_{i=1}^{N} x_i^2 / \sigma_i^2}$$

For uniform uncertainty $\sigma_i^2 = 1.5$ for all y_i
$(\sigma_i^2 = \sigma_y^2)$

$$b = \frac{+\sum x_i y_i}{\sum x_i^2} = 3.60 \quad \text{via Matlab}$$

$$\left(\begin{array}{l} S_{xx} = \sum_i x_i \cdot x_i \\ S_{xy} = \sum_i x_i \cdot y_i \end{array} \right)$$

$$\sigma_b^2 = \sum_{i=1}^{N} \left(\frac{\partial b}{\partial y_i}\right)^2 \sigma_{y_i}^2$$

only $i = l$ is relevant

$$\frac{\partial b}{\partial y_l} = \frac{\partial}{\partial y_l} \frac{\sum_{i=1}^{N} x_i y_i / \sigma_i^2}{\sum x_i^2 / \sigma_i^2} = \frac{x_l / \sigma_l^2}{\sum x_i^2 / \sigma_i^2}$$

If $\sigma_i^2 = \sigma_y^2$ is constant for all points

$$b = \frac{\frac{1}{\sigma_y^2} \sum x_i y_i}{\frac{1}{\sigma_y^2} \sum x_i^2} = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$\sigma_b^2 = \sum_{i=1}^{N} \left(\frac{\partial b}{\partial y_i}\right)^2 \sigma_{y_i}^2 \qquad \frac{\partial b}{\partial y_l} = \frac{\partial}{\partial y_l} \frac{(x_1 y_1 + x_2 y_2 + \cdots x_N y_N)}{x_1^2 + x_2^2 + \cdots x_N^2}$$

$$\Big\downarrow \underset{\text{const}}{\sigma_{y_i}^2 = \sigma_y^2} \qquad \frac{\partial b}{\partial y_l} = \frac{x_l}{\sum x_i^2}$$

$$= \sigma_y^2 \sum_{l=1}^{N} \left(\frac{\partial b}{\partial y_l}\right)^2$$

$$= \sigma_y^2 \sum_{l=1}^{N} \left(\frac{x_l}{\underbrace{\sum_i x_i^2}_{\text{constant}}}\right)^2 = \frac{\sigma_y^2}{(S_{xx})^2} \sum_{l=1}^{N} x_l^2 = \frac{\sigma_y^2}{S_{xx}^2} S_{xx}$$

$$\sigma_b^2 = \frac{\sigma_y^2}{S_{xx}} = \frac{(1.5)^2}{\underset{\text{(via matlab)}}{2600}} = 8.65 \times 10^{-4}$$

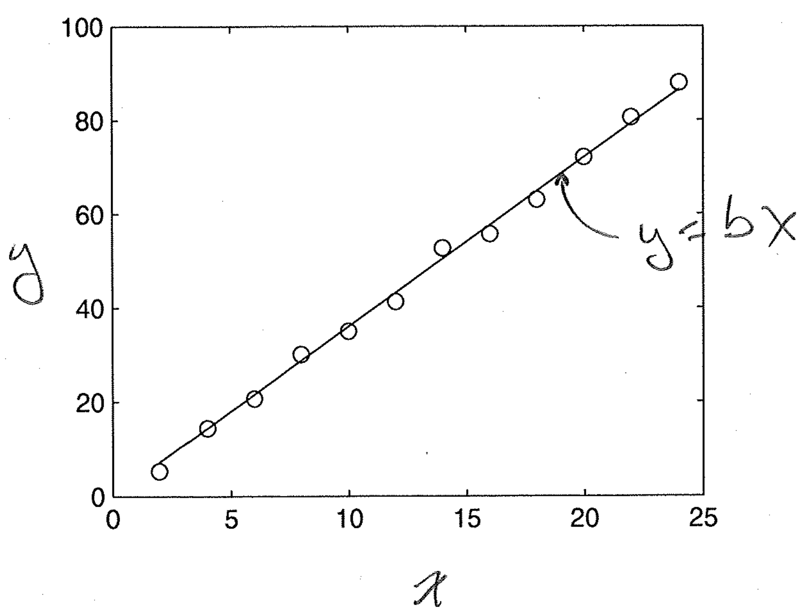
$$\sigma_b = 0.029 \Rightarrow b = 3.60 \pm 0.03$$

$$\chi^2 = \frac{1}{\sigma_y^2} \sum (y_i - b x_i)^2 = 11.9 \quad \text{via Matlab}$$

$\approx$ # of points

```
x = [2 4 6 8 10 12 14 16 18 20 22 24 ]
x =
     2     4     6     8    10    12    14    16    18    20    22    24
>> y = [ 5.3 14.4 20.7 30.1 35.0 41.3 52.7 55.7 63.0 72.1 80.5 87.9 ] ;
>> Sxy = sum( x .* y )
Sxy =
       9364.4
>> Sxx = sum( x .* x )
Sxx =
         2600
>> b = Sxy / Sxx
b =
       3.6017
>> sigmay =1.5; w = 1. / (sigmay^2);
>> chisqr = w * sum( (y .- b*x).^2 )
 chisqr = w * sum( (y .- b*x).^2 )
                        |
Error: Unexpected MATLAB operator.

>> chisqr = w * sum( (y - b*x).^2 )
chisqr =
       11.912
```



#4 Least squares fit to linearized free-fall expt

t (s)	0	0.05	0.10	0.15	0.20	0.25	0.30
x (cm)	0.0	9.7	21.9	36.7	53.6	73.0	94.7
x/t ($\frac{cm}{s}$)	—	194.0	219.	245.	268.	292	316.

$$x = v_0 t + \tfrac{1}{2} a t^2$$

$$\frac{x}{t} = v_0 + \left(\frac{a}{2}\right) t$$

$$(\ y = a + bx\ )$$

Assuming equal weights ($\sigma_i^2 = \sigma_y^2$ & $W_i = 1/\sigma_y^2$)

Via <u>Bevington</u>: (uniform σ_y^2)

$$a = \frac{1}{\Delta}\frac{1}{\sigma_y^4}\left[S_{x^2} S_y - S_x S_{xy}\right]$$

with $\Delta = \frac{1}{\sigma_y^4}\left[N S_{x^2} - (S_x)(S_{\cancel{x}})\right]$

and $b = \frac{1}{\Delta}\left[\frac{N}{\sigma_y^4} S_{xy} - \frac{1}{\sigma_y^4} S_x S_y\right]$

$$= \frac{1}{\sigma_y^4}\frac{1}{\Delta}\left[N S_{xy} - S_x S_y\right]$$

Recall $S_? = \sum_{i=1}^{N} (?)_i$, $? = x_i, x_i y_i, x_i^2, y_i$

so $$a = \frac{S_{x^2} S_y - S_x S_{xy}}{N S_{x^2} - (S_x)^2}$$

$$b = \frac{N S_{xy} - S_x S_y}{N S_{x^2} - (S_x)^2}$$

Sums found

$a =$ [illegible] $\frac{cm}{s}$ $\qquad b =$ [illegible] $\frac{cm}{s^2}$ Via Matlab See code

$$\sigma_y^2 \chi^2 = \sum_{i=1}^{N} [y_i - (a+bt_i)]^2 \qquad \left(y_i = \frac{x_i}{t_i} \text{ here}\right)$$

For a correctly weighted ($1/\sigma_y^2$) sch we expect $\chi^2 \cong N-2 = \nu$ = degrees of freedom
\# of points − \# of parameters - slope, intercept
$\Rightarrow 6-2 = 4$ here

$$\sum (y_i - (a+bt_i))^2 = 3.28 \text{ via Matlab}$$

We then estimate $\sigma_y^2 \approx \frac{3.3}{4} \approx 0.82 \qquad \sigma_y \approx 0.9$

This is the uncertainty in our x/t not x itself!

```
>> t = [ 0.05 0.10 0.15 0.20 0.25 0.30 ];
>> y = [ 194. 219. 245. 268. 292. 316. ];
>> Sx = sum(t)

Sx =

    1.0500

>> compact
Undefined function or variable 'compact'.

>> display compact
compact
>> Sxx = sum( t .* t); Sxy = sum(t .*y); Sy = sum(y);
>> N=6;
>> delta = N*Sxx - Sx*Sx;
>> a = (Sxx*Sy - Sx*Sxy) / delta

a =

  170.4667

>> b = (N*Sxy - Sx*Sy) / delta

b =

  486.8571

>> chisqr = sum( (y - (a + b*t)).^2 )

chisqr =

    3.2762
```

$(= x/t)$

$S_x = \sum t_i$

$S_{xy} = \sum y_i t_i$

$S_{xx} = \sum t_i^2$

$x_i \rightarrow t_i$

$y = a + bx$

$\left(\frac{x}{t} = V_0 + \frac{a}{2}t\right)$

a: $\left(\frac{cm}{s}\right)$

b: $\left(\frac{cm}{s^2} = \frac{1}{2}\text{ acceleration}\right)$