2 Background

The tuning fork is a mechanical oscillator. The tuning fork is made from quartz crystal, which has piezoelectric properties. In this section we will review the behavior of a mechanical oscillator and a series LCR circuit and point out the analogy between the mechanical and electrical systems. The LCR circuit is useful since the piezoelectric behavior of quartz is used to electrically measure the mechanical motion of the harmonic oscillator.

**Driven harmonic oscillator**

\[
\begin{align*}
\ddot{x} + k x - b \dot{x} + F_0 e^{i \omega t} = 0
\end{align*}
\]

Figure 2: Simple harmonic oscillator.

Consider a simple model of a mechanical oscillator with a spring of spring constant \(k\) fixed at one end and a mass \(m\) attached to the other end, driven by an external driving force \(F(t) = F_0 e^{i \omega t}\). Applying Newton’s 2\textsuperscript{nd} law, \(m \ddot{x} = \Sigma F\), to the above system results in

\[
m \ddot{x} = -k x - b \dot{x} + F_0 e^{i \omega t},
\]  

(1)
where \( b \) is the viscous damping constant. Eq. 1 can be written as

\[
\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t},
\]

where \( \gamma = \frac{b}{m} \) and \( \omega_0^2 = \frac{k}{m} \) is the resonant frequency. Let the solution of Eq. 2 be \( x(t) = A e^{i\omega t} \). With \( \dot{x}(t) = iA e^{i\omega t} \) and \( \ddot{x}(t) = -\omega^2 x(t) \), Eq. 2 can be written as

\[
A e^{i\omega t} (\omega_0^2 + i\gamma\omega - \omega^2) = \frac{F_0}{m} e^{i\omega t}.
\]

Solving for \( A \),

\[
A = \frac{f}{(\omega_0^2 + i\gamma\omega - \omega^2)},
\]

where \( f = \frac{F_0}{m} \) and \( A \) describes the frequency dependent response to the driving force.

Let us introduce the quality factor, represented as \( Q \), defined as the ratio of resonant frequency and damping \( \frac{\omega_0}{\gamma} \). We can write the frequency response in terms of \( \omega_0 \), \( \omega \) and \( Q \) as

\[
A = \frac{f}{(\omega_0^2 - \omega^2) + i\frac{\omega_0}{Q}}.
\]

\( A \) has real and imaginary terms. For simplicity we can represent the frequency dependent amplitude as \( A(\omega) = |A| \), so we can write

\[
A(\omega) = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\frac{\omega_0}{Q})^2}}.
\]

(3)

The phase shift \( \delta \) is the shift in the phase of displacement with respect to the driving force. The phase shift is found from the real and imaginary parts.
of the denominator in the frequency response term,

\[ \delta = \tan^{-1} \left[ \frac{\omega_0}{Q(\omega_0^2 - \omega^2)} \right]. \]

Figure 3 is a plot of frequency dependent amplitude squared as a function of \( \omega \) plotted with \( Q \) values 100 and 150. We can see the resonant behavior when driving frequency \( \omega \) is close to the natural frequency \( \omega_0 \). Also we can see how the amplitude increases with the increase in the quality factor of the oscillator.
The quality factor $Q$ of the harmonic oscillator can also be found by calculating the resonance width at half maximum. If $\Delta \omega$ is the resonance width at half maximum (figure 4), the quality factor is

$$Q = \frac{\omega_0}{\Delta \omega}.$$  

![Diagram](image)

Figure 4: $\Delta \omega$ represents the resonance width at half maximum.

The displacement is given as $x(t) = Ae^{i\omega t}$. In terms of $A(\omega)$ and $\delta$ the
The displacement is

\[ x(t) = A(\omega)e^{i(\omega t - \delta)} \]

\[ x(t) = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\frac{\omega_0}{Q})^2}} e^{i(\omega t - \delta)}. \tag{4} \]

**LCR circuit and its behavior**

The model LCR circuit consists of inductor (L), capacitor (C), and a resistor (R) connected in series and driven with voltage \( V_0e^{i\omega t} \). Applying Kirchhoff’s law, \( \Sigma V = V_0e^{i\omega t} \), around the loop

![Series LCR circuit](image)

Figure 5: Series LCR circuit.
\[
\frac{q}{C} + iR + L \frac{di}{dt} = V_0 e^{i\omega t},
\]

where \( q \) is the charge on the capacitor, \( i = \frac{dq}{dt} = \dot{q} \) and \( \frac{di}{dt} = \ddot{q} \). Then Eq. 4 can be written as

\[
\frac{1}{LC} q + \frac{R}{L} \frac{dq}{dt} + \frac{1}{L} \frac{d^2q}{dt^2} = \frac{V_0}{L} e^{i\omega t}.
\]

Setting the square of resonant frequency \( \omega_0^2 = \frac{1}{LC} \) and damping \( \gamma = \frac{R}{L} \), the above equation can be written as

\[
\omega_0^2 q + \gamma \frac{dq}{dt} + \frac{1}{L} \frac{d^2q}{dt^2} = \frac{V_0}{L} e^{i\omega t}.
\]

Comparing Eq. 2 and Eq. 6 we can see that there is an analogy between the mechanical resonant system and the electrical resonant system. The table below gives us the classification of the analogous quantities in a mechanical system and an electrical system.

<table>
<thead>
<tr>
<th>Mechanical system</th>
<th>Electrical system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement ((x))</td>
<td>Charge ((q))</td>
</tr>
<tr>
<td>Driving force ((F))</td>
<td>Driving voltage ((V))</td>
</tr>
<tr>
<td>Mass ((m))</td>
<td>Inductance ((L))</td>
</tr>
<tr>
<td>Viscous force constant ((b))</td>
<td>Resistance ((R))</td>
</tr>
<tr>
<td>Spring constant ((k))</td>
<td>Reciprocal capacitance ((\frac{1}{C}))</td>
</tr>
<tr>
<td>Resonant frequency (\sqrt{\frac{L}{m}})</td>
<td>Resonant frequency (\frac{1}{\sqrt{LC}})</td>
</tr>
<tr>
<td>Damping constant ((\gamma=\frac{b}{m}))</td>
<td>Damping constant ((\gamma=\frac{R}{L}))</td>
</tr>
</tbody>
</table>

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From the table above we can see that the displacement in a mechanical resonant system is same as the charge in an electrical resonant system. When the tuning fork is driven with some drive voltage, the arms of the tuning fork vibrate and current is produced. The mechanical motion of the harmonic oscillator can be monitored by measuring the current in the tuning fork. The simple way of interpreting this current is using impedances in the LCR circuit.

The total impedance of the LCR circuit is given as

\[ Z = Z_L + Z_C + Z_R, \]  

(7)

where \( Z_L = i\omega L \) is the impedance of the inductor, \( Z_C = \frac{1}{i\omega C} \) is the impedance of the capacitor, and \( Z_R = R \) is the impedance of the resistor. \( Z \) is then

\[ Z = i\omega L + \frac{1}{i\omega C} + R \]

and the magnitude of \( Z \) is

\[ |Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}. \]

Using \( \omega_0^2 = \frac{1}{LC} \) and \( \gamma = \frac{R}{L} = \frac{\omega_0}{Q} \), the above equation can be written as

\[ |Z| = \left(\frac{L}{\omega}\right)\sqrt{\left(\frac{\omega\omega_0}{Q}\right)^2 + (\omega^2 - \omega_0^2)^2}. \]

The admittance \( Y \), which is the inverse of impedance, has a magnitude of

\[ |Y| = Y(\omega) \text{ given by} \]
\[
Y(\omega) = \frac{\omega}{L\sqrt{\left(\frac{\omega_0}{\omega}\right)^2 + (\omega^2 - \omega_0^2)^2}}.
\]  

(8)

Eq. 3 and Eq. 8 are similar (with \(L\) analogous to \(m\)) other than there is an additional \(\omega\) in the numerator of the admittance term. The effect of this extra \(\omega\) is negligible on the shape of the resonance near \(\omega_0\). If we plot \(Y^2(\omega)\) as function of \(\omega\), the plot is similar to the graph of mechanical amplitude as a function of frequency.

![Figure 6: The square of admittance as a function of frequency with \(Q=100\) (solid) and \(Q=150\) (dashed)](image_url)

Figure 6 is a plot of admittance as a function of \(\omega\) plotted with \(Q\) values...
100 and 150. We can see the resonant behavior when driving frequency $\omega$ is close to the natural frequency $\omega_0$. Also we can see how the amplitude increases with the increase in the quality factor of the oscillator.

The phase shift $\delta$ in the electrical system is the shift in the phase of current with respect to drive voltage. The phase shift is found from the real and imaginary parts of the denominator in the admittance term,

$$\delta = \tan^{-1}\left[\frac{\omega_0}{Q(\omega_0^2 - \omega^2)}\right].$$

The current in the LCR circuit is

$$I = \frac{V}{Z}, \quad (9)$$

$$I = VY. \quad (10)$$

Using Eq. 8, the current in the LCR circuit in terms of phase shift, drive voltage and admittance is,

$$I(t) = \frac{V_0\omega}{L\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega_0/Q)^2}}e^{i(\omega t - \delta)}. \quad (11)$$

Eq. 4 gives the displacement of a harmonic oscillator and Eq. 11 gives the current in the LCR circuit. These equations are similar but we see an extra $\omega$ in Eq. 8 because we are taking the time derivative of charge in finding the total current in a LCR circuit.
Later when we analyze the data we will use the square of the admittance. Because of the involvement of real and imaginary parts in the admittance term, it is more convenient to use the square of admittance.

Since quartz, the material of the tuning forks, has piezoelectric properties as described in the next section, the mechanical motion of the tuning fork can be sensed electrically. The series LCR circuit turns out to be a convenient way to describe the oscillating tuning fork.
3 Quartz tuning forks

Some crystals, such as quartz, exhibit a relationship between mechanical strain and voltage across their surfaces. When compressed or stretched, these crystals will build up opposite charges on the faces, thus acting like a capacitor with an applied voltage. When these crystals are subjected to an external voltage, the crystals expand or contract accordingly.\textsuperscript{4}

The quartz tuning forks are normally used in wrist watches. They are commercially available in a closed can, with 2 electrodes connected to the faces of the arms of tuning fork. They have a resonant frequency of \(32768=2^{15}\) Hz. The dimensions\textsuperscript{5} of one arm of the tuning fork are \(3.6 \times .6 \times .25\) mm\(^3\).

When we drive the quartz tuning fork with an oscillating external drive voltage the arms of the tuning fork vibrate as the arms are expanding and contracting. Opposite charges are produced on the faces of the arms of tuning fork, and the amount of charge produced is proportional to the displacement of the arms of tuning fork, \(\delta x \propto q\).

When the arms of the tuning fork are vibrating, current is produced, so the mechanical motion of the tuning fork can be monitored by measuring the current in the tuning fork.

The convenient way of interpreting this piezoelectric behavior is to model the mechanical harmonic oscillator with the LCR circuit analog. This model is improved by adding a stray capacitance in parallel to the LCR circuit. The stray capacitance\textsuperscript{6} \(C_0\) is mainly due to wiring and the capacitor formed by the electrodes on the tuning fork.
The admittance of this LCR circuit with a stray capacitance in parallel is calculated by adding the admittance of $C_0$ to the admittance of LCR circuit.

If $Y_1$ is the admittance of the LCR circuit and $Y_2$ is the admittance of $C_0$. The total admittance $Y_N$, is

$$Y_N = \frac{1}{i\omega L + \frac{1}{i\omega C} + R} + i\omega C_0.$$  

The magnitude $|Y_N| = Y_N(\omega)$, is

$$Y_N(\omega) = \sqrt{\frac{R^2\omega^2C_0^2C^2 + (C + C_0 - \omega^2LCC_0)^2}{R^2 + (\omega L - \frac{1}{\omega C})^2}}.$$  

Setting $\omega_p^2 = \frac{1}{LC_0}$ and $\omega_0^2 = \frac{1}{LC}$, the magnitude of the admittance in terms of $\omega_p$, $\omega$, $\omega_0$ and $Q$ is
\[ Y_N(\omega) = \frac{\omega}{L \omega_p} \left( \sqrt{\left(\frac{\omega_p}{Q}\right)^2 + \left(\omega_p^2 + \omega_0^2 - \omega^2\right)^2} \right) \frac{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{\omega_p}{Q}\right)^2}}{\sqrt{\omega_0^2 - \omega^2}}. \] (12)

Figure 8 is a plot of \( Y_N^2 \) as a function of \( \omega \) plotted with \( Q \) values 100, 150. We again can see the resonant behavior when driving frequency is close to the resonant frequency, and as in the case of LCR circuit the admittance peak increases with increase in \( Q \) value. Also we can see some asymmetry in the shape of the curve because of the stray capacitance.

Figure 8: \( Y_N \) as a function of \( \omega \) with \( Q =100 \) (solid) and 150 (dashed).
Current measurement

We can measure the current in the tuning fork by building an op-amp based current-to-voltage converter shown in Figure 9. The tuning fork is driven by a signal generator, resulting in a piezoelectric current \( I(t) = Y_N V_0 e^{i\omega t} \), which must also flow through the feedback resistor \( R_f \).

\[
V_{\text{out}}(t) = -R_f V_0 e^{i\omega t} Y_N. \quad (13)
\]

In terms of phase shift and admittance, \( V_{\text{out}}(t) \) is

\[
V_{\text{out}}(t) = -\frac{\omega}{L} (\sqrt{\frac{(\omega_0 \delta)^2 + (\omega_p^2 + \omega_0^2 - \omega^2)^2}{(\omega_0^2 - \omega_p^2)^2 + (\omega_0 \delta)^2}}) R_f V_0 e^{i(\omega t - \delta)}. \]

Figure 9: Circuit to measure current in the tuning fork.
Lock-in amplifier and its application in the experiment

A lock-in amplifier,\(^8\) is used to measure the in-phase and quadrature components of a signal. It is called a “lock-in” because it locks to and measures the signal at particular frequency of interest, ignoring all other signals at the input. The essential part of the lock-in amplifier is a phase sensitive detector. The output of lock-in-amplifier is a function of the relative phase angle between the input signal and an associated reference signal. The lock-in amplifier can also be used to measure the relative phase relationship of two signals of the same frequency.

A lock-in requires 2 inputs, signal \(S(t)\) and reference \(R(t)\) which is usually the drive signal of the experiment. The main function of a lock-in depends on multiplying the \(S(t)R(t)\) and filtering the unwanted signals.

\[ S(t) = S_0 \cos(\omega t - \delta) \]

\[ R(t) = \cos(\omega t - \delta_r) \]

Figure 10: Lock-in operation.

Let \(S(t) = S_0 \cos(\omega t - \delta)\) be the signal and \(R(t) = \cos(\omega t - \delta_r)\) be the reference.
Expanding $S(t) = S_0 \cos(\omega t - \delta)$,

$$S(t) = S_0 \cos \delta \cos \omega t + S_0 \sin \delta \sin \omega t.$$  \hspace{1cm} (14)

Let $S_i = S_0 \cos \delta$ and $S_q = S_0 \sin \delta$, and Eq. 11 becomes

$$S(t) = S_i \cos \omega t + S_q \sin \omega t.$$  \hspace{1cm} (15)

These two terms are described as the in-phase component and quadrature component ($\frac{1}{4}$ cycle out of phase with the drive signal $V_D \cos \omega t$) respectively.

The output of the mixer is

$$V_{mix}(t) = S(t)R(t)$$  \hspace{1cm} (16)

Expanding Eq. 15

$$V_{mix}(t) = \frac{1}{2} S_i [\cos \delta_r + \cos(2\omega t - \delta_r)] + \frac{1}{2} S_q [\cos (-\frac{\pi}{2} + \delta_r) + \cos(2\omega t - \delta_r)].$$  \hspace{1cm} (17)

The low pass filter eliminates the high frequency ($2\omega t$) terms.

If $\delta_r = 0$, Eq. 16 becomes just

$$V_{filt} = \frac{1}{2} S_i.$$  \hspace{1cm} (18)

If $\delta_r = \frac{\pi}{2}$, Eq. 16 becomes

$$V_{filt} = \frac{1}{2} S_q.$$  \hspace{1cm} (19)

Therefore with $\delta_r = 0$ the lock-in measures the in-phase component of $S(t)$ and with $\delta_r = \frac{\pi}{2}$ it measures the quadrature component of $S(t)$.  

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So the lock-in will give us the two parts of the signal, \( X = S_i \) and \( Y = S_q \), now

\[
S(t) = S_i(t) + S_q(t),
\]
or for amplitudes,

\[
S^2 = S_i^2 + S_q^2.
\]

For the tuning fork (LCR circuit with \( C_0 \) in parallel) the signal measured is the voltage proportional to a current \( I(t) \)

\[
S(t) = -I(t)R_f = -\frac{\omega}{2\pi \rho} \left( \sqrt{\left(\frac{\omega_0}{Q}\right)^2 + (\omega_p^2 + \omega_0^2 - \omega^2)^2} \right) \frac{R_f V_0 e^{i\omega t - \delta}}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\frac{\omega_0}{Q})^2}}.
\]

The quadrature and in-phase terms add up and give us the total current in the tuning fork.