Cosines and Cayley, Triangles and Tetrahedra

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Abstract. This article surveys some of the more aesthetically appealing and useful formulas relating distances, areas, and angles in triangles and tetrahedra. For example, a somewhat neglected trigonometric identity involving only the cosines of a triangle is an instance of the famous Cayley cubic surface. While most of these formulas are well known, some novel identities also make an appearance.

Heron's formula and the Cayley–Menger determinant. Many people learn in school that for a triangle with vertices A, B, and C, the area of the triangle, Δ_{ABC}, can be computed from Heron's formula [**2, 13**],

$$\Delta_{ABC} = \sqrt{s(s - r_{AB})(s - r_{AC})(s - r_{BC})} \tag{1}$$

in which r_{ij} is the distance between vertex i and vertex j, and s is the semiperimeter $s = \dfrac{r_{AB} + r_{AC} + r_{BC}}{2}$.

Almost two thousand years later, a form of Heron's formula was found that generalizes to simplices of any dimension. This is the Cayley–Menger determinant; for a triangle:

$$\begin{aligned} -16\Delta_{ABC}^2 &= \det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & r_{AB}^2 & r_{AC}^2 \\ 1 & r_{AB}^2 & 0 & r_{BC}^2 \\ 1 & r_{AC}^2 & r_{BC}^2 & 0 \end{pmatrix} \\ &= -(-r_{AB} + r_{AC} + r_{BC})(r_{AB} - r_{AC} + r_{BC})(r_{AB} + r_{AC} - r_{BC}) \\ &\quad \times (r_{AB} + r_{AC} + r_{BC}). \end{aligned}$$

The matrix in the determinant is called the Cayley–Menger matrix. Cayley found the determinantal form [**7**]—the polynomial itself was known earlier, by Lagrange. Menger discovered a number of further properties of the matrix and closely related variants of it [**14**]. For an $n-1$-dimensional simplex of n vertices $A_1, \dots A_n$, the volume formula generalizes to

$$\Delta_{A_1\dots A_n}^2 = \frac{(-1)^{n+1}}{2^n (n!)^2} \det \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & r_{A_1A_2}^2 & \dots & r_{A_1A_n}^2 \\ 1 & r_{A_1A_2}^2 & 0 & \dots & r_{A_2A_n}^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & r_{A_1A_n}^2 & r_{A_2A_n}^2 & \dots & 0 \end{pmatrix}. \tag{2}$$

The definitive article about the Cayley–Menger matrix is [**5**], and its properties are also nicely summarized in [**4**]. The use of distances as coordinates is masterfully

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investigated in [**1**] and, with particular attention paid to triangles, in [**8**]. Some uses and variants of the Cayley–Menger determinant are discussed in [**11, 12**].

Laws of cosines. For brevity in some of the more complex formulae, we write c_{ABC} for the cosine of the angle ABC.

The classic law of cosines is

$$r_{AB}^2 - r_{AC}^2 - r_{BC}^2 + 2r_{AC}r_{BC}c_{ACB} = 0.$$

One derivation of the law of cosines makes use of the relations of the form

$$r_{AB} = r_{AC}c_{BAC} + r_{BC}c_{ABC} \tag{3}$$

which can be used for many purposes because of their linearity in the distances and cosines. These relations generalize to the n-dimensional simplex through the polyhedral version of the divergence theorem. If we choose a vector field to be the outward-pointing normal $\vec{n}_i$ to the ith facet of a bounded convex polyhedron, then the divergence of the field is 0. This is equal to the surface integral

$$0 = \sum_j \vec{n}_i \cdot \vec{n}_j \Delta_j = \Delta(i) - \sum c_{ij}\Delta_j \tag{4}$$

where c_{ij} is the cosine of the angle between facets i and j, and Δ_i is the area of the ith facet. Multiplying this equation by Δ_i and subtracting Δ_j times the corresponding equation for face j gives

$$\Delta_i^2 = \sum_{j \neq i} \Delta_j^2 - 2 \sum_{j,k \neq i} c_{jk}\Delta_j\Delta_k. \tag{5}$$

While equation (5) has the same form as the law of cosines for triangles, it seems much less useful since more than one cosine is involved in each equation. The lower degree conditions in (4) are usually a better starting point for other identities.

A neglected trigonometric identity: the Cayley cosine cubic. If we think of the equations (3) as linear in the distances, then the distances must be in the kernel of the coefficient matrix, i.e.,

$$\begin{pmatrix} -1 & c_{BAC} & c_{ABC} \\ c_{BAC} & -1 & c_{ACB} \\ c_{ABC} & c_{ACB} & -1 \end{pmatrix} \begin{pmatrix} r_{AB} \\ r_{AC} \\ r_{BC} \end{pmatrix} = 0.$$

If we choose to normalize the kernel vector by setting $r_{AB} = 1$, then

$$r_{AC} = \frac{c_{BAC} + c_{ACB}c_{ABC}}{1 - c_{ACB}^2}, \quad \text{and}$$

$$r_{BC} = \frac{c_{ABC} + c_{ACB}c_{BAC}}{1 - c_{ACB}^2}$$

which are complementary to the law of cosines in that they express a single distance in terms of the cosines. Of course, these formulae break down if the triangle is collinear.

In order to have a nontrivial kernel, the determinant must vanish, which gives us the beautiful relation:

$$\det \begin{pmatrix} -1 & c_{BAC} & c_{ABC} \\ c_{BAC} & -1 & c_{ACB} \\ c_{ABC} & c_{ACB} & -1 \end{pmatrix} = c_{ACB}^2 + c_{ABC}^2 + c_{BAC}^2 + 2c_{ACB}c_{ABC}c_{BAC} - 1 = 0. \quad (6)$$

This trigonometric identity has been known for a long enough time that it is difficult to determine its first appearance. But it does not seem well known, and it is not usually cited as an example of the famous Cayley cubic, a surface with the maximal number (four) of isolated singular points (where both the surface and the gradient of its defining function vanish). Perhaps we should call it the Cayley cosine cubic.

The four singular points of the Cayley cosine cubic are

$$(c_{123}, c_{132}, c_{213}) \in \{(1, 1, -1), (1, -1, 1), (-1, 1, 1), (-1, -1, -1)\}.$$

The first three of these points correspond to collinear triangles, while the last one is unrealizable as a triangle. However, it is consistent algebraically with a triangle whose edge lengths add up to zero ($r_{AB} + r_{AC} + r_{BC} = 0$). Intriguingly, this is the fourth factor in the Cayley–Menger determinant as well.

The Cayley cosine cubic contains six lines that intersect the planes $c_{ijk} = \pm 1$. The three lines with $c_{ijk} = 1$ are the boundary of the portion of the surface corresponding to triangles with positive distances.

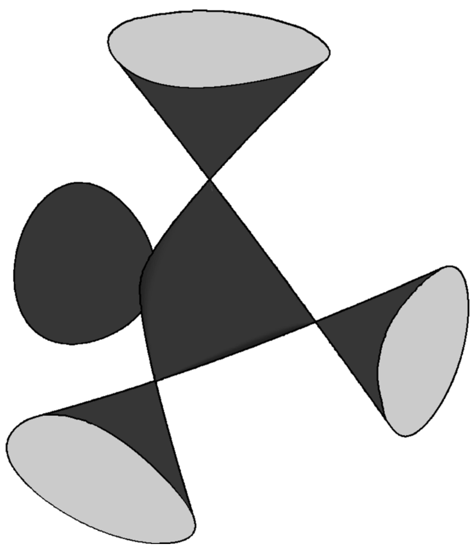

Figure 1. The Cayley cubic.

In a tetrahedron, these laws of cosines generalize in a variety of ways. There is a Cayley cosine cubic for each of the four faces, involving the three facial cosines (cosines between the edges of the tetrahedron).

The Cayley cosine cubic determinant (6) generalizes directly to the dihedral angles of a tetrahedron [**3**], where c_{ij} is the cosine of the angle between faces i and j:

$$\det\begin{pmatrix} -1 & c_{AB} & c_{AC} & c_{AD} \\ c_{AB} & -1 & c_{BC} & c_{BD} \\ c_{AC} & c_{BC} & -1 & c_{CD} \\ c_{AD} & c_{BD} & c_{CD} & -1 \end{pmatrix} = 0.$$

This formula may relate to a rigidity result [**15**] on tetrahedra, which states that if all of the dihedral angles of a tetrahedron are less than or equal to the corresponding angles of another tetrahedron, then the two tetrahedra must be similar.

The dihedral angles can be computed in terms of facial cosines at a vertex. For each dihedral angle, there are two choices for the vertex. For instance, the cosine of the dihedral angle between faces A and B can be expressed as

$$c_{AB} = \frac{c_{ADB} - c_{ADC}c_{BDC}}{s_{ADC}s_{BDC}} = \frac{c_{ACB} - c_{BCD}c_{ACD}}{s_{BCD}s_{ACD}}$$

from which we can eliminate the sines by squaring and substituting the Pythagorean identity to obtain a relation between the facial cosines:

$$\begin{aligned}(1 - c_{ACD}^2)(1 - c_{BCD}^2)(c_{ADB} - c_{ADC}c_{BDC})^2 \\ = (1 - c_{ADC}^2)(1 - c_{BDC}^2)(c_{ACB} - c_{BCD}c_{ACD})^2. \end{aligned} \tag{7}$$

For each of the six edges of the tetrahedron, we have such an equation. These cannot be independent since the space of similarity classes of tetrahedra is five-dimensional.

Somewhat similarly, the law of sines can be used once for each triangular face to obtain the identity

$$s_{ACB}s_{BDC}s_{CAD}s_{ABD} = s_{BAC}s_{CBD}s_{ACD}s_{ADB},$$

which could be converted into a cosine identity by squaring both sides. This is somewhat unsatisfactory, however, since it involves eight angles.

If a geometric problem can be cast into polynomial form, the computation of a Gröbner basis (or bases) provides an automated path for eliminating variables and obtaining new or simpler relations [**6**] (*caveat emptor*: in practice, many Gröbner bases require excessive memory and computational time to compute). It is beyond the scope of this article to describe Gröbner bases in full. They are analogous to the reduction of a linear system to echelon form (Gaussian elimination), but for polynomial (nonlinear) systems. For more background on Gröbner bases, see [**9**].

By computing a Gröbner basis for the system of equations (4) for a tetrahedron (using Singular [**10**]), we found a fairly simple condition on the six angles bordering one face. The six angles are on three faces but do not include any of the three angles meeting at the common vertex. For a common vertex A, this condition is

$$\begin{aligned} &c_{ACB}^2c_{ADC}^2c_{ABD}^2 - c_{ADB}^2c_{ABC}^2c_{ACD}^2 + c_{ADB}^2c_{ABC}^2 - c_{ACB}^2c_{ADC}^2 \\ &\quad - c_{ACB}^2c_{ABD}^2 - c_{ADC}^2c_{ABD}^2 + c_{ADB}^2c_{ACD}^2 + c_{ABC}^2c_{ACD}^2 + c_{ACB}^2 \\ &\quad - c_{ADB}^2 - c_{ABC}^2 + c_{ADC}^2 + c_{ABD}^2 - c_{ACD}^2 = 0. \end{aligned}$$

If we use the squares as variables, i.e., let $q_{ijk} = c_{ijk}^2$, then there are nine singular planes corresponding to collinear configurations, along with the origin. This singular point at the origin could be interpreted as a projective closure having point A at infinity.

This can be written in a nicer form:

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & c_{ABC}^2 & c_{ABC}^2 \\ 1 & c_{ACD}^2 & 1 & c_{ACD}^2 \\ 1 & c_{ADB}^2 & c_{ADB}^2 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & c_{ACB}^2 & c_{ACB}^2 \\ 1 & c_{ADC}^2 & 1 & c_{ADC}^2 \\ 1 & c_{ABD}^2 & c_{ABD}^2 & 1 \end{pmatrix}. \tag{8}$$

This determinantal form was found by an *ad hoc* approach. Is there an elementary way to derive this identity? We do not know of one, but it seems likely.

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