Classify the following ODEs as linear, separable, and/or degree-homogeneous, and indicate their order. If possible, find the general solution to each (in some cases only an implicit solution is possible).

1. \( x \frac{dy}{dx} + xy = 1 - y. \)

   Solution: This is a linear first-order equation. The integrating factor \( \mu = xe^x, \) and the general solution is \( y = Ce^{-x}/x + 1/x. \)

2. \( (e^x + 1) \frac{dy}{dx} = y - ye^x. \)

   Solution: This is linear and separable. The easiest way to do the hardest integral that appears (\( \int \frac{1-e^x}{1+e^x} dx \)) is to use the hyperbolic trig functions \( \cosh(x) = (e^x + e^{-x})/2 \) and \( \sinh(x) = (e^x - e^{-x})/2, \) although it is also possible just using the u-substitution \( u = 1 + e^x. \) In terms of \( \cosh \) the solution is \( y = C/(\cosh(x/2))^2, \) or \( y = \frac{4C}{e^x + e^{-x} + 2}. \)

3. \( \frac{dy}{dx} = e^{x+y}. \)

   Solution: This is separable, since \( e^{x+y} = e^x e^y. \) After separating and integrating, we get \( -e^{-y} = e^x + C. \) Solving for \( y \) we get \( y = \ln(-e^x - C). \) Every solution blows up after some finite \( x \) interval since the logarithm’s argument will become zero.

4. \( \frac{dy}{dx} = \frac{x^2 + y^2}{x^2}. \)

   Solution: This is degree-homogeneous so we use the substitution \( v = y/x, \) or \( vx = y. \) Differentiating this gives us \( y' = v + xv'. \) Substituting these relations gives a new ODE for \( v, v + xv' = 1 + v^2, \) or \( v'(v^2 - v + 1) = 1/x. \) This is separable. The left-hand integral is somewhat difficult; after a linear change of variables \( u = (2\sqrt{3}v - \sqrt{3})/3 \) it becomes an arctan integral and we get

   \[
   \frac{2\sqrt{3}}{3} \arctan\left(\frac{2v\sqrt{3} - \sqrt{3}}{3}\right) = \ln(x) + C.\]

   After substituting \( v = y/x \) back in, this becomes the implicit solution

   \[
   \frac{2\sqrt{3}}{3} \arctan\left(\frac{2y\sqrt{3} - x\sqrt{3}}{3x}\right) = C + \ln(x)
   \]

   which can be solved for \( y \) to get

   \[
   y = \sqrt{3}x \tan\left(\frac{\sqrt{3}(C + \ln(x))}{2}\right) / 2 + x/2.
   \]