This is longer than the actual exam, which will be 8 to 10 questions (some might be multiple choice). You are allowed up to two sheets of notes (both sides) and a calculator, although any use of a calculator must be indicated.

(1) Find the general solution to \((1 + t)y' + y = \cos t\).

(2) Rewrite the initial value problem \(y''' + y'' + y = t\), \(y(0) = y'(0) = y''(0) = 0\) as an equivalent first-order system.

(3) Find the general solution to the system
\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

(4) Are the vectors \(v_1 = (1, 2, 3, 4)\), \(v_2 = (2, -2, 4, 2)\), and \(v_3 = (0, -3, -1, -3)\) linearly independent? If not, write one of them as a linear combination of the other two.

(5) Solve the initial value problem \(y'' + y = \cos x\), \(y'(0) = 0\), \(y(0) = -\frac{1}{2}\).

(6) Use Euler’s, the Improved Euler’s, or the Runge-Kutta method to numerically approximate \(y(2)\) to two digits of accuracy if \(y' = t + \sqrt{y}\) and \(y(0) = 1\).

(7) Find the general solution to the system
\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

(8) Find the Laplace transform \(X(s) = \mathcal{L}(x(t))\) if \(x'' + 8x' + 15x = 0\) and \(x(0) = 0\), \(x'(0) = 1\). Then find the solution \(x(t)\).

(9) What is the \textbf{form} of the general solution to the ODE \(y''' - 4y'' + 14y' - 20y = te^t \cos (3t) + t^2\). Hint: one of the roots of the characteristic polynomial of the left-hand side is 2. For extra credit find the values of the constants in the particular solution.
(10) Consider a mass-spring system with two masses of mass \( m_1 \) and \( m_2 \). Mass 1 is connected to a wall with a spring of stiffness \( k_1 \) and to mass 2 with a spring of stiffness \( k_2 \). Mass 2 is connected to a second wall with a spring of stiffness \( k_3 \), as shown below. Their displacements from the equilibrium are \( x_1 \) and \( x_2 \), which we will combine into a vector \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). Then if \( x'' = Ax \), show that the eigenvalues of \( A \) must be negative if the masses and spring constants are positive.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{mass_spring.png}
\caption{Mass-spring system diagram}
\end{figure}

(11) Use either the Laplace transform method or the eigenvalue/eigenvector method to find the steady state solution to the initial value problem \( x' = -x - z, \ y' = -x - y, \ z' = 2x + z, \ x(0) = 0, \ y(0) = 0, \ z(0) = 2 \).

(12) Find the equilibria of the system \( x' = 2y^3 - 2x, \ y' = x^2 - 1 \), and determine their stability by computing the eigenvalues of the linearized systems.

(13) Three identical, well-stirred tanks of with 100 liters of water in each tank are connected in series with tank 1 pumping 10 liter/minute into tank 2, tank 2 pumping 10 liter/minute into tank 3, and tank 3 pumping 10 liter/minute into tank 1. If tank 1 initially has 500 grams of salt dissolved in it, and the other two tanks start at time \( t = 0 \) with no salt, which of the following initial value problems describes the amounts of salt in grams in each tank (\( x_1 = \) salt in tank 1, \( x_2 = \) salt in tank 2, \( x_3 = \) salt in tank 3).

(a) \[ x'_1 = \frac{1}{10} x_2 - \frac{1}{10} x_1 \quad x'_2 = \frac{1}{10} x_1 - \frac{1}{10} x_2 \quad x'_3 = \frac{1}{10} x_2 - \frac{1}{10} x_3 \]
\[ x_1(0) = 500, \ x_2(0) = 0, \ x_3(0) = 0. \]

(b) \[ x'_1 = \frac{1}{100} x_2 - \frac{1}{10} x_1 \quad x'_2 = \frac{1}{100} x_3 - \frac{1}{10} x_2 \quad x'_3 = \frac{1}{100} x_1 - \frac{1}{10} x_3 \]
\[ x_1(0) = 500, \ x_2(0) = 0, \ x_3(0) = 0. \]
(c) \[ x'_1 = \frac{1}{10} x_3 - \frac{1}{10} x_1 \quad x'_2 = \frac{1}{10} x_1 - \frac{1}{10} x_2 \quad x'_3 = \frac{1}{10} x_2 - \frac{1}{10} x_3 \]

\[ x_1(0) = 500, \ x_2(0) = 0, \ x_3(0) = 0. \]

(d) \[ x'_1 = \frac{1}{10} x_3 + \frac{1}{10} x_1 \quad x'_2 = \frac{1}{10} x_1 + \frac{1}{10} x_2 \quad x'_3 = \frac{1}{10} x_2 + \frac{1}{10} x_3 \]

\[ x_1(0) = 500, \ x_2(0) = 0, \ x_3(0) = 0. \]

(14) (a) Find the recurrence relation for the power series solution around \( x = 0 \) for the differential equation

\[ \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 4\alpha y = 0. \]

(b) Using your result from part (a), find the solution to the initial value problem

\[ \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 8y = 0, \ y(0) = 1, \ y'(0) = 0. \]