Math 3280 Practice Final Solutions

This is longer than the actual exam, which will be 8-10 questions. You are allowed up to two sheets of notes (both sides) and a calculator, although any use of a calculator must be indicated. Please let me know if you find any typos in these solutions.

(1) Find the general solution to $(1+t)y' + y = \cos t$.

Solution: In standard form ($y' + P(t)y = Q(t)$) we have $y' + \frac{1}{1+t}y = \frac{\cos(t)}{1+t}$. Using the integrating factor method (section 1.5), we have

$$\rho(t) = e^{\int P(t)dt} = e^{\log(1+t)} = 1 + t.$$

Then $\int \rho\ Q\ dt = \int \cos t\ dt = \sin t$ and

$$y = \frac{C}{\rho} + \frac{1}{\rho}\int \rho Q dt = \frac{C}{1+t} + \frac{\sin t}{1+t}.$$

(2) Rewrite the initial value problem $y''' + y'' + y = t$, $y(0) = y'(0) = y''(0) = 0$ as an equivalent first-order system.

Solution: Introduce the variables $v_1 = y'$, $v_2 = v_1' = y''$ and the system becomes:

$$\begin{aligned} y' &= v_1 \\ v_1' &= v_2 \\ v_2' &= t - v_2 - y \\ y(0) = 0,\ v_1(0) &= 0,\ v_2(0) = 0 \end{aligned}$$

Note that rewriting the initial conditions is a required part of this answer.

(3) Find the general solution to the system

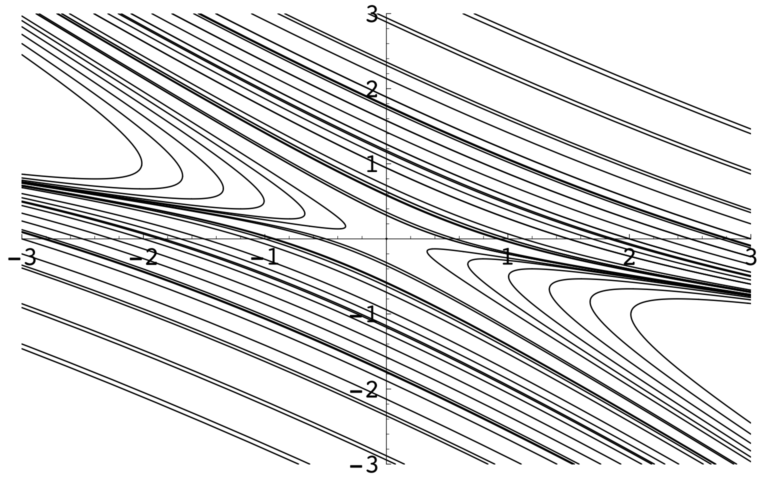
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The eigenvalues of the matrix are found from

$$det \begin{bmatrix} 2-\lambda & 4 \\ -1 & -3-\lambda \end{bmatrix} = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$$

From row-reducing $A - \lambda I$ for each of these two eigenvalues ($\lambda = 1$ and $\lambda = -2$) we can find that the eigenvectors are $\vec{v_1} = (-4, 1)$ and $\vec{v_2} = (-1, 1)$, so the solutions are $x_1 = -4C_1e^t - C_2e^{-2t}$ and $x_2 = C_1e^t + C_2e^{-2t}$.

For large t, $(x_1, x_2) \approx e^t(-4C_1, C_1)$. For large $-t$, $(x_1, x_2) \approx e^{-2t}(-C_2, C_2)$. Some trajectories are shown below.



(4) Are the vectors $v_1 = (1,2,3,4)$, $v_2 = (2,-2,4,2)$, and $v_3 = (0,-3,-1,-3)$ linearly independent? If not, write one of them as a linear combination of the other two.

Solution: The vectors are linearly dependent if there are c_1, c_2, c_3, not all zero, such that $c_1v_1 + c_2v_2 + c_3v_3 = 0$. This is equivalent to the coefficient matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & -3 \\ 3 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix}$ having less than 3 pivots after row-reduction. If we row-reduce A we find

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & -3 \\ 3 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -6 & -3 \\ 0 & -2 & -1 \\ 0 & -6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This only has two pivots. The free variable is c_3, which we can choose to be 2 (to avoid fractions - it would be OK to set it to any nonzero value), which gives $c_2 = -1$ and $c_1 = 2$.

So $2v_1 - v_2 + 2v_3 = 0$; we can write any of the vectors in terms of the other two but the easiest choice here is $v_2 = 2v_1 + 2v_3$.

(5) Solve the initial value problem $y'' + y = \cos x$, $y'(0) = 0$, $y(0) = -\frac{1}{2}$.

Solution: This could be also done with a Laplace transform. Using undetermined coefficients we find the solution by decomposing it into $y = y_h + y_p$. The homogeneous solution y_h is found from the characteristic equation $r^2 + 1 = (r-i)(r+i) = 0$ to be $y_h = C_1\cos(x) + C_2\sin(x)$.

Since the right-hand side $\cos(x)$ is contained in the solution space of the homogeneous equation, we are forced to consider particular solutions of the form $y_p = Ax\cos(x) + Bx\sin(x)$. Then $y_p'' = -Ax\cos(x) - 2A\sin(x) + 2B\cos(x) - Bx\sin(x)$. Substituting these forms into our ODE yields $-2A\sin(x) + 2B\cos(x) = \cos(x)$, so $A = 0$ and $B = 1/2$.

So now we know that $y = C_1\cos(x) + C_2\sin(x) + x\sin(x)/2$. Evaluating this using the initial conditions we get $C_2 = 0$ and $C_1 = -\frac{1}{2}$, so $y = \dfrac{-\cos(x) + x\sin(x)}{2}$.

(6) Use Euler's, the Improved Euler's, or the Runge-Kutta method to numerically approximate $y(2)$ to two digits of accuracy if $y' = t + \sqrt{y}$ and $y(0) = 1$.

Solution: It takes 76 steps to get the desired accuracy with Euler's Method (so this is somewhat harder than anything I would require on the actual final exam). For the improved Euler's method, 5 steps are needed. Fourth-order Runge-Kutta works in 1 step (stepsize 2), giving $y(2) \approx 6.37$ which agrees with $y(2) = 6.411474127809772838513\ldots$ in the first two digits after rounding:

$$f(x,y) = x + \sqrt{y}, \quad h = 2, \quad x_0 = 0, \quad y_0 = 1$$

$$k_1 = f(x_0, y_0) = f(0,1) = 1$$

$$k_2 = f(x_0 + h/2, y_0 + hk_1/2) = f(1,2) = 1 + \sqrt{2} \approx 2.41421356$$

$$k_3 = f(x_0 + h/2, y_0 + hk_2/2) = f(1, \sqrt{2}+2) = \sqrt{\sqrt{2}+2} + 1 \approx 2.84775907$$

$$k_4 = f(x_0 + h, y_0 + hk_3) = f(2, 2\sqrt{\sqrt{2}+2}+3) = \sqrt{2\sqrt{\sqrt{2}+2}+3} + 2 \approx 4.58756993$$

$$y(2) \approx y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \sqrt{\sqrt{2}+2} + \frac{1}{3}\sqrt{2\sqrt{\sqrt{2}+2}+3} + \frac{2}{3}\sqrt{2} + \frac{10}{3} \approx 6.37050506$$

It is not necessary to keep the intermediate calculations in exact form, as done above, but you do need to be careful to include enough digits to avoid rounding error.

(7) Find the general solution to the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The characteristic equation is $\det(A - \lambda I) = \lambda^2 - 4\lambda + 8$ with roots (eigenvalues) $\lambda = 2 \pm 2i$. We need to find one eigenvector, lets find it for $\lambda = 2 + 2i$. We now row reduce

$$A - (2+2i)I = \begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \to \begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}$$

So the eigenvector can be chosen to be $v = (-1+2i, 1)$. Then the solution to the system is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 Re[ve^{2t}(\cos(2t) + i\sin(2t))] + C_2 Im[ve^{2t}(\cos(2t) + i\sin(2t))]$$
$$= C_1 \begin{bmatrix} -e^{2t}(\cos(2t) + 2\sin(2t)) \\ e^{2t}\cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} e^{2t}(-\sin(2t) + 2\cos(2t)) \\ e^{2t}\sin(2t) \end{bmatrix}$$

(8) Find the Laplace transform $X(s) = \mathcal{L}(x(t))$ if $x'' + 8x' + 15x = 0$ and $x(0) = 0$, $x'(0) = 1$. Then find the solution $x(t)$.

Solution: Taking the Laplace transform of the ODE gives

$$s^2X(s) + 8sX(s) + 15X(s) - 8x(0) - sx(0) - x'(0)$$
$$= s^2X(s) + 8sX(s) + 15X(s) - 1 = 0.$$

Solving for $X(s)$ and performing a partial fraction decomposition, we get

$$X(s) = \frac{1}{s^2 + 8s + 15} = \frac{1/2}{s+3} - \frac{1/2}{s+5}$$

Since $\mathcal{L}^{-1}(\frac{1}{s-a}) = e^{at}$, we can invert $X(s)$ to get $x(t) = \frac{e^{-3t}}{2} - \frac{e^{-5t}}{2}$.

The use of the Laplace transform will sometimes be optional on the final, but for problems with initial conditions such as $x(0) = 0$, $x'(0) = 0$ it can be the easiest method.

(9) What is the **form** of the general solution to the ODE $y''' - 4y'' + 14y' - 20y = te^t\cos(3t) + t^2$. Hint: one of the roots of the characteristic polynomial of the left-hand side is 2.

Solution:

First we find the homogeneous solution. The characteristic equation can be factored using the hint to get

$$r^3 - 4r^2 + 14r - 20 = (r-2)\left(r^2 - 2r + 10\right)$$

and then we can use the quadratic equation to get $r = 2, 1 \pm 3i$. So the homogeneous solutions is $y_h = C_1e^t\sin(3t) + C_2e^t\cos(3t) + C_3e^{2t}$.

If there were no overlap with the homogeneous solution we would use the form

$$Ate^t\cos(3t) + Bte^t\sin(3t) + Ce^t\cos(3t) + De^t\sin(3t) + Et^2 + Ft + G$$

for the particular solution, but the terms with C and D are contained in the homogeneous solution so we multiply everything involving this root (the A,B,C, and D terms) by t to get the form of the particular solution:

$$y_p = At^2e^t\cos(3t) + Bt^2e^t\sin(3t) + Cte^t\cos(3t) + Dte^t\sin(3t) + Et^2 + Ft + G.$$

The form of the general solution is the sum of these, $y = y_h + y_p$.

(10) Consider a mass-spring system with two masses of mass m_1 and m_2. Mass 1 is connected to a wall with a spring of stiffness k_1 and to mass 2 with a spring of stiffness k_2. Mass 2 is a connected to a second wall with a spring of stiffness k_3, as shown below. Their displacements from the equilibrium are x_1 and x_2, which we will combine into a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then if $x'' = Ax$, show that the eigenvalues of A must be negative if the masses and spring constants are positive.

Solution: As discussed in chapter 7.4, the matrix A has the form:

$$\begin{pmatrix} -(k_1+k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -(k_2+k_3)/m_2 \end{pmatrix}$$

One way to see that the eigenvalues are negative if the k_i and m_i are positive is to use the fact that if λ_1 and λ_2 are the eigenvalues of A, then $tr(A) = \lambda_1 + \lambda_2$ and $det(A) = \lambda_1\lambda_2$. The determinant can be simplified to

$$det(A) = \frac{k_1k_2 + k_1k_3 + k_2k_3}{m_1m_2}$$

which is clearly positive. This means the eigenvalues must have the same sign, but since their trace is negative the sign of each must be negative.

(11) Use either the Laplace transform method or the eigenvalue/eigenvector method to find the steady state solution to the initial value problem $x' = -x - z$, $y' = -x - y$, $z' = 2x + z$, $x(0) = 0$, $y(0) = 0$, $z(0) = 2$.

Solution: Using the eigenvalue/eigenvector method we first compute the eigenvalues of the coefficient matrix $A = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ from the characteristic equation $det(A - \lambda I) = 0$. This factors as $(\lambda+1)(\lambda^2+1) = 0$, so the eigenvalues are $\pm i$ and -1.

Now we find the eigenvectors. For $\lambda = -1$ we row reduce

$$A + I = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

to get $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The kernel of this matrix consists of vectors of the form $\begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$ for any a. We can choose $a = 1$.

For the complex conjugate pair we can use either eigenvalue. If we choose to use i, then we row reduce $A - iI$ to get $\begin{pmatrix} 1 & 0 & 1/2 - i/2 \\ 0 & 1 & i/2 \\ 0 & 0 & 0 \end{pmatrix}$. If we choose the last entry of the eigenvalue to be 2, the eigenvector is $\begin{pmatrix} -1+i \\ -i \\ 2 \end{pmatrix}$.

The general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + C_2 Re(\begin{pmatrix} -1+i \\ -i \\ 2 \end{pmatrix} (\cos(t) + i\sin(t))) + C_3 Im(\begin{pmatrix} -1+i \\ -i \\ 2 \end{pmatrix} (\cos(t) + i\sin(t)))$$

$$= C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -\cos(t) - \sin(t) \\ \sin(t) \\ 2\cos(t) \end{pmatrix} + C_3 \begin{pmatrix} \cos(t) - \sin(t) \\ -\cos(t) \\ 2\sin(t) \end{pmatrix}.$$

Now we can use the initial conditions; evaluating at $t = 0$ gives $-C_2 + C_3 = 0$, $C_1 - C_3 = 0$, and $2C_2 = 2$. So $C_3 = 1$ and $C_1 = 1$. For the steady state solution we drop the first term since e^{-t} will decay to 0. So the steady state solution is:

$$\begin{pmatrix} -2\sin(t) \\ -\cos(t) + \sin(t) \\ 2\cos(t) + 2\sin(t) \end{pmatrix}$$

(12) Find the equilibria of the system $x' = 2y^3 - 2x$, $y' = x^2 - 1$, and determine their stability by computing the eigenvalues of the linearized systems.

Solution: To find the equilibria we solve the pair of equations $2y^3 - 2x = 0$, $x^2 - 1 = 0$. The second equation is simpler, since it only involves x - any equilibria must have $x = \pm 1$. Substituting these values into the first equation gives $y^3 = \pm 1$, so $y = \pm 1$ and y is the same sign as x. I.e. the two equilibria are $(1, 1)$ and $(-1, -1)$.

The Jacobian matrix of the functions $f_1(x, y) = 2y^3 - 2x$ and $f_2 = x^2 - 1$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial(2y^3 - 2x)}{\partial x} & \frac{\partial(2y^3 - 2x)}{\partial y} \\ \frac{\partial(x^2 - 1)}{\partial x} & \frac{\partial(x^2 - 1)}{\partial y} \end{pmatrix} = \begin{pmatrix} -2 & 6y^2 \\ 2x & 0 \end{pmatrix}$$

At the equilibrium $(1, 1)$ this becomes $\begin{pmatrix} -2 & 6 \\ 2 & 0 \end{pmatrix}$. The eigenvalues are solutions of $\lambda^2 + 2\lambda - 12 = 0$, which are $-1 \pm \sqrt{13}$. Since $-1 + \sqrt{13} > 0$, there is a positive eigenvalue and the equilibrium is unstable.

At the equilibrium $(-1, -1)$ the Jacobian becomes $\begin{pmatrix} -2 & 6 \\ -2 & 0 \end{pmatrix}$. The eigenvalues are solutions of $\lambda^2 + 2\lambda + 12 = 0$, which are $-1 \pm \sqrt{11}i$. Since the real parts of these are negative, the equilibrium is stable (nearby solutions would spiral inwards).

(13) Three identical, well-stirred tanks of with 100 liters of water in each tank are connected in series with tank 1 pumping 10 liter/minute into tank 2, tank 2 pumping 10 liter/minute into tank 3, and tank 3 pumping 10 liter/minute into tank 1. If tank 1 initially has 500 grams of salt dissolved in it, and the other two tanks start at time $t = 0$ with no salt, which of the following initial value problems describes the amounts of salt in grams in each tank (x_1 = salt in tank 1, x_2 = salt in tank 2, x_3 = salt in tank 3).

Solution: Answer (c) is correct

$x_1' = \frac{1}{10}x_3 - \frac{1}{10}x_1 \qquad x_2' = \frac{1}{10}x_1 - \frac{1}{10}x_2 \qquad x_3' = \frac{1}{10}x_2 - \frac{1}{10}x_3$

(14) (a) Find the recurrence relation for the power series solution around $x = 0$ for the differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 4\alpha y = 0.$$

Solution:

We assume that $y = \sum_{n=0}^{\infty} c_n x^n$. Then the differential equation becomes

$$\sum_{n=0}^{\infty}(c_{n+2}(n+2)(n+1) - 2nc_n + 4\alpha c_n)x^n = 0$$

with recurrence relation

$$c_{n+2} = \frac{2(n-2\alpha)c_n}{(n+2)(n+1)}$$

(b) Using your result from part (a), find the solution to the initial value problem $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 8y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution:

The initial conditions imply that $c_0 = 1$ and $c_1 = 0$. Thus only the even terms of the series are nonzero. From the recurrence relation, with $\alpha = 2$, we have $c_2 = -4$, $c_4 = 4/3$, and $c_6 = 0$. Since c_6 is zero, all terms after it are also zero, and the series is actually the polynomial $y = 1 - 4x^2 + 4x^4/3$.