(1) Approximate $y(1)$ if $y' = x^2 + y^2$ and $y(0) = 1$ using Euler's method with 4 steps. Compare that approximation to that given by the improved Euler method with 2 steps, and the Runge-Kutta method using 1 step. Which is the best approximation?

Solution: Euler's method with 4 steps has $(x_1, y_1) = (1/4, 5/4)$, $(x_2, y_2) = (1/2, 53/32)$, $(x_3, y_3) = (3/4, 9849/4096)$, and $(x_4, y_4) = (1, 267806001/67108864)$. This gives the approximation $y(1) \approx 267806001/67108864 = 3.99\ldots$

The improved Euler's method with 2 steps has $(x_1, y_1) = (1/2, 15/8)$ and $(x_2, y_2) = (1, 432321/65536)$ which gives $y(1) \approx y_2 = 432321/65536 = 6.5967\ldots$

One step of the fourth-order Runga-Kutta gives $y(1) \approx y_1 = 16249/1536 = 10.578776041\bar{6}$.

If you continued to use more and more accurate methods, you would find your approximation continued to grow. For example, 200 steps of Euler's method gives $y(1) \approx y_{200} = 1502$. In fact this solution does not exist at $x = 1$, it blows up (goes to $+\infty$) near $x = 0.97$. (This is example 5 in Chapter 2.4 in the text.)

(2) Solve the initial value problem $\frac{dy}{dx} = 3\frac{y}{x} - 3x^5$, $y(2) = 56$.

Solution: After putting this in standard first-order linear form, we can see that $P(x) = -3/x$ and $Q(x) = -3x^5$. Then the integrating factor is $\rho = e^{-3\ln(x)} = x^{-3}$ and

$$y = C/\rho + \frac{1}{\rho}\int \rho Q \, dx = Cx^3 + x^3(-x^3) = Cx^3 - x^6$$

Using the initial condition we can find C: $56 = C(2)^3 - 2^6$, so $C = 120/8 = 15$. So finally: $y(x) = 15x^3 - x^6$.

(3) Sherlock Holmes is awoken by a phone call from a policeman at 3:30am. A body has been discovered and foul play is suspected. Sherlock tells the police to determine the temperature of the body and, when he arrives at the scene 45 minutes later, he takes the temperature again. The two readings that cold 60 degree F morning were 80 degrees F and 70 degrees F. When was the latest time that the body was 98.6? (Note that in this problem, the readings are taken far enough apart that you should not use a slope estimate to determine the parameter in the ODE.)

Solution: the differential equation is Newton's law of cooling: $T' = k(A - T)$, which has the solution

$$T(t) = A + (T_0 - A)e^{-k(t-t_0)}$$

if $T(t_0) = T_0$.

If we choose to consider $t_0 = 0$ at 3:30 am, and use t in minutes, then $T_0 = 80$ and we can first determine k from the condition

$$T(45) = 70 = 60 + (80 - 60)e^{-45k}$$

After solving for $k = -\ln(.5)/45 \approx 0.0154$, we can solve for the value of t which gives $98.6 = 60 + 20e^{-kt}$. This gives $t = -42.7$ so the time of death was about 2:47 am.

(4) The model $y' = -ky$ is a little too simple to provide a good model for some pharmocokinetic phenomena, such as the concentration of alcohol in the blood after having a drink. To be more accurate we can use a two-tank model.

Let tank 1 model the stomach, with a volume of 1 liter and an initial concentration of 2% alcohol. Assume fluid is absorbed into the bloodstream (tank 2) from the stomach at a rate of 100 milliliters per minute, and that the volume of the stomach stays constant (i.e. replaced without additional alcohol). The volume of the blood is 5 liters, initially with no alcohol, and fluid is exchanged between the blood and the body tissues at a rate of 100 milliliters per minute. You can assume that all alcohol transferred into the body is metabolized, so none of it returns to the bloodstream. Find the maximum concentration of alcohol in the bloodstream.

Solution: Let x_1 and x_2 denote the amounts of alcohol in the two compartments respectively.

Then our equations are:

$$x_1' = -x_1/10, \qquad x_1(0) = 0.02$$

$$x_2' = x_1/10 - x_2/50, \qquad x_2(0) = 0$$

The first equation can be solved (by reference, or using its linearity, or its separability) to obtain $x_1 = 0.02e^{-t/10}$.

This can be substituted into the differential equation for x_2, and then re-arranged into standard form:

$$x_2' + x_2/50 = 0.002e^{-t/10}$$

We will use the linear solution

$$x_2 = \frac{C}{\rho} + \frac{1}{\rho}\int \rho Q dt$$

The integrating factor is $\rho = e^{t/50}$. After integrating

$$\int \rho Q dt = 0.002 \int e^{t/50}e^{-t/10}dt = -0.025e^{-0.08t}$$

and using the initial condition to determine the constant, we obtain the solution for x_2:

$$x_2 = 0.025(e^{-0.02t} - e^{-t/10})$$

To find the maximum we differentiate x_2, set the derivative equal to zero, and solve for t:

$$t = \frac{-\ln(1/10)}{.08} \approx 20.12 \text{ minutes}$$

Then we can evaluate

$$x_2(20.12) \approx 0.014$$

So the maximum concentration in the blood is $0.01337/5 = 0.267\%$.