

## Math 3280 Practice Final Solutions

This is longer than the actual exam, which will be 7-8 questions. You are allowed up to two sheets of notes (both sides) and a calculator, although any use of a calculator must be indicated. Please let me know if you find any typos in these solutions.

- (1) Find the general solution to  $(1+t)y' + y = \cos t$ .

Solution: In standard form  $(y' + P(t)y = Q(t))$  we have  $y' + \frac{1}{1+t}y = \frac{\cos(t)}{1+t}$ . Using the integrating factor method (section 1.5), we have

$$\rho(t) = e^{\int P(t)dt} = e^{\log(1+t)} = 1+t.$$

Then  $\int \rho Q dt = \int \cos t dt = \sin t$  and

$$y = \frac{C}{\rho} + \frac{1}{\rho} \int \rho Q dt = \frac{C}{1+t} + \frac{\sin t}{1+t}.$$

- (2) Rewrite the initial value problem  $y''' + y'' + y = t$ ,  $y(0) = y'(0) = y''(0) = 0$  as an equivalent first-order system.

Solution: Introduce the variables  $v_1 = y'$ ,  $v_2 = v_1' = y''$  and the system becomes:

$$\begin{aligned} y' &= v_1 \\ v_1' &= v_2 \\ v_2' &= t - v_2 - y \\ y(0) &= 0, \quad v_1(0) = 0, \quad v_2(0) = 0 \end{aligned}$$

Note that rewriting the initial conditions is a required part of this answer.

- (3) Find the general solution to the system

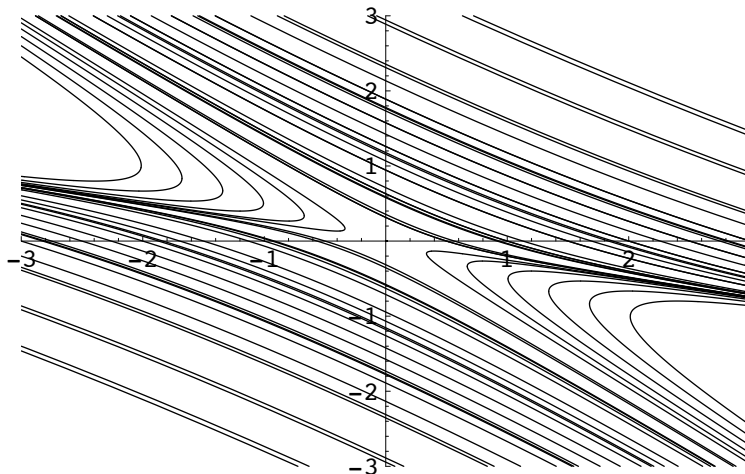
$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The eigenvalues of the matrix are found from

$$\det \begin{bmatrix} 2-\lambda & 4 \\ -1 & -3-\lambda \end{bmatrix} = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$$

From row-reducing  $A - \lambda I$  for each of these two eigenvalues ( $\lambda = 1$  and  $\lambda = -2$ ) we can find that the eigenvectors are  $\vec{v}_1 = (-4, 1)$  and  $\vec{v}_2 = (-1, 1)$ , so the solutions are  $x_1 = -4C_1e^t - C_2e^{-2t}$  and  $x_2 = C_1e^t + C_2e^{-2t}$ .

For large  $t$ ,  $(x_1, x_2) \approx e^t(-4C_1, C_1)$ . For large  $-t$ ,  $(x_1, x_2) \approx e^{-2t}(-C_2, C_2)$ . Some trajectories are shown below.



- (4) Are the vectors  $v_1 = (1, 2, 3, 4)$ ,  $v_2 = (2, -2, 4, 2)$ , and  $v_3 = (0, -3, -1, -3)$  linearly independent? If not, write one of them as a linear combination of the other two.

Solution: The vectors are linearly dependent if there are  $c_1, c_2, c_3$ , not all zero, such that  $c_1v_1 + c_2v_2 + c_3v_3 = 0$ . This is equivalent to the coefficient

matrix  $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & -3 \\ 3 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix}$  having less than 3 pivots after row-reduction. If

we row-reduce  $A$  we find

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & -3 \\ 3 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -6 & -3 \\ 0 & -2 & -1 \\ 0 & -6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This only has two pivots. The free variable is  $c_3$ , which we can choose to be 2 (to avoid fractions - it would be OK to set it to any nonzero value), which gives  $c_2 = -1$  and  $c_1 = 2$ .

So  $2v_1 - v_2 + 2v_3 = 0$ ; we can write any of the vectors in terms of the other two but the easiest choice here is  $v_2 = 2v_1 + 2v_3$ .

- (5) Solve the initial value problem  $y'' + y = \cos x$ ,  $y'(0) = 0$ ,  $y(0) = -\frac{1}{2}$ .

Solution: This could be also done with a Laplace transform. Using undetermined coefficients we find the solution by decomposing it into  $y = y_h + y_p$ .

The homogeneous solution  $y_h$  is found from the characteristic equation  $r^2 + 1 = (r - i)(r + i) = 0$  to be  $y_h = C_1 \cos(x) + C_2 \sin(x)$ .

Since the right-hand side  $\cos(x)$  is contained in the solution space of the homogeneous equation, we are forced to consider particular solutions of the form  $y_p = Ax \cos(x) + Bx \sin(x)$ . Then  $y_p'' = -Ax \cos(x) - 2A \sin(x) + 2B \cos(x) - Bx \sin(x)$ . Substituting these forms into our ODE yields  $-2A \sin(x) + 2B \cos(x) = \cos(x)$ , so  $A = 0$  and  $B = 1/2$ .

So now we know that  $y = C_1 \cos(x) + C_2 \sin(x) + x \sin(x)/2$ . The initial conditions become  $C_2 = 0$  and  $C_1 = -\frac{1}{2}$ , so  $y = \frac{-\cos(x) + x \sin(x)}{2}$ .

- (6) Use Euler's, the Improved Euler's, or the Runge-Kutta method to numerically approximate  $y(2)$  to two digits of accuracy if  $y' = t + \sqrt{y}$  and  $y(0) = 1$ .

Solution: It takes 76 steps to get the desired accuracy with Euler's Method (so this is somewhat harder than anything I would require on the actual final exam). For the improved Euler's method, 5 steps are needed. Fourth-order Runge-Kutta works in 1 step (stepsize 2), giving  $y(2) \approx 6.37$  which agrees with  $y(2) = 6.411474127809772838513 \dots$  in the first two digits after rounding:

$$f(x, y) = x + \sqrt{y}, \quad h = 2, \quad x_0 = 0, \quad y_0 = 1$$

$$k_1 = f(x_0, y_0) = f(0, 1) = 1$$

$$k_2 = f(x_0 + h/2, y_0 + hk_1/2) = f(1, 2) = 1 + \sqrt{2} \approx 2.41421356$$

$$k_3 = f(x_0 + h/2, y_0 + hk_2/2) = f(1, \sqrt{2} + 2) = \sqrt{\sqrt{2} + 2} + 1 \approx 2.84775907$$

$$k_4 = f(x_0 + h, y_0 + hk_3) = f(2, 2\sqrt{\sqrt{2} + 2} + 3) = \sqrt{2\sqrt{\sqrt{2} + 2} + 3} + 2 \approx 4.58756993$$

$$y(2) \approx y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \sqrt{\sqrt{2} + 2} + \frac{1}{3} \sqrt{2\sqrt{\sqrt{2} + 2} + 3} + \frac{2}{3} \sqrt{2} + \frac{10}{3} \approx 6.37050506$$

It is not necessary to keep the intermediate calculations in exact form, as done above, but you do need to be careful to include enough digits to avoid rounding error.

- (7) Find the general solution to the system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The characteristic equation is  $\det(A - \lambda I) = \lambda^2 - 4\lambda + 8$  with roots (eigenvalues)  $\lambda = 2 \pm 2i$ . We need to find one eigenvector, let's find it for

$\lambda = 2 + 2i$ . We now row reduce

$$A - (2 + 2i)I = \begin{bmatrix} -1 - 2i & -5 \\ 1 & 1 - 2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 - 2i \\ 0 & 0 \end{bmatrix}$$

So the eigenvector can be chosen to be  $v = (-1 + 2i, 1)$ . Then the solution to the system is

$$\begin{aligned} x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= C_1 \operatorname{Re}[ve^{2t}(\cos(2t) + i \sin(2t))] + C_2 \operatorname{Im}[ve^{2t}(\cos(2t) + i \sin(2t))] \\ &= C_1 \begin{bmatrix} -e^{2t}(\cos(2t) + 2 \sin(2t)) \\ e^{2t} \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} e^{2t}(-\sin(2t) + 2 \cos(2t)) \\ e^{2t} \sin(2t) \end{bmatrix} \end{aligned}$$

- (8) Find the Laplace transform  $X(s) = \mathcal{L}(x(t))$  if  $x'' + 8x' + 15x = 0$  and  $x(0) = 0$ ,  $x'(0) = 1$ . Then find the solution  $x(t)$ .

Solution: Taking the Laplace transform of the ODE gives

$$\begin{aligned} s^2 X(s) + 8sX(s) + 15X(s) - 8x(0) - sx(0) - x'(0) \\ = s^2 X(s) + 8sX(s) + 15X(s) - 1 = 0. \end{aligned}$$

Solving for  $X(s)$  and performing a partial fraction decomposition, we get

$$X(s) = \frac{1}{s^2 + 8s + 15} = \frac{1/2}{s + 3} - \frac{1/2}{s + 5}$$

Since  $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$ , we can invert  $X(s)$  to get  $x(t) = \frac{e^{-3t}}{2} - \frac{e^{-5t}}{2}$ .

The use of the Laplace transform will sometimes be optional on the final, but for problems with initial conditions such as  $x(0) = 0$ ,  $x'(0) = 0$  it can be the easiest method.

- (9) What is the **form** of the general solution to the ODE  $y''' - 4y'' + 14y' - 20y = te^t \cos(3t) + t^2$ . Hint: one of the roots of the characteristic polynomial of the left-hand side is 2.

Solution:

First we find the homogeneous solution. The characteristic equation can be factored using the hint to get

$$r^3 - 4r^2 + 14r - 20 = (r - 2)(r^2 - 2r + 10)$$

and then we can use the quadratic equation to get  $r = 2, 1 \pm 3i$ . So the homogeneous solutions is  $y_h = C_1 e^t \sin(3t) + C_2 e^t \cos(3t) + C_3 e^{2t}$ .

If there were no overlap with the homogeneous solution we would use the form

$$Ate^t \cos(3t) + Bte^t \sin(3t) + Ce^t \cos(3t) + De^t \sin(3t) + Et^2 + Ft + G$$

for the particular solution, but the terms with  $C$  and  $D$  are contained in the homogeneous solution so we multiply everything involving this root (the  $A, B, C$ , and  $D$  terms) by  $t$  to get the form of the particular solution:

$$y_p = At^2e^t \cos(3t) + Bt^2e^t \sin(3t) + Cte^t \cos(3t) + Dte^t \sin(3t) + Et^2 + Ft + G.$$

The form of the general solution is the sum of these,  $y = y_h + y_p$ .

- (10) Consider a mass-spring system with two masses of mass  $m_1$  and  $m_2$ . Mass 1 is connected to a wall with a spring of stiffness  $k_1$  and to mass 2 with a spring of stiffness  $k_2$ . Mass 2 is connected to a second wall with a spring of stiffness  $k_3$ , as shown below. Their displacements from the equilibrium are  $x_1$  and  $x_2$ , which we will combine into a vector  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ . Then if  $x'' = Ax$ , show that the eigenvalues of  $A$  must be negative if the masses and spring constants are positive.

Solution: As discussed in chapter 7.4, the matrix  $A$  has the form:

$$\begin{pmatrix} -(k_1 + k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -(k_2 + k_3)/m_2 \end{pmatrix}$$

One way to see that the eigenvalues are negative if the  $k_i$  and  $m_i$  are positive is to use the fact that if  $\lambda_1$  and  $\lambda_2$  are the eigenvalues of  $A$ , then  $\text{tr}(A) = \lambda_1 + \lambda_2$  and  $\det(A) = \lambda_1\lambda_2$ . The determinant can be simplified to

$$\det(A) = \frac{k_1k_2 + k_1k_3 + k_2k_3}{m_1m_2}$$

which is clearly positive. This means the eigenvalues must have the same sign, but since their trace is negative the sign of each must be negative.

- (11) Use either the Laplace transform method or the eigenvalue/eigenvector method to find the steady state solution to the initial value problem  $x' = -x - z$ ,  $y' = -x - y$ ,  $z' = 2x + z$ ,  $x(0) = 0$ ,  $y(0) = 0$ ,  $z(0) = 2$ .

Solution: Using the eigenvalue/eigenvector method we first compute the eigenvalues of the coefficient matrix  $A = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$  from the characteristic equation  $\det(A - \lambda I) = 0$ . This factors as  $(\lambda + 1)(\lambda^2 + 1) = 0$ , so the eigenvalues are  $\pm i$  and  $-1$ .

Now we find the eigenvectors. For  $\lambda = -1$  we row reduce

$$A + I = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

to get  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ . The kernel of this matrix consists of vectors of the form  $\begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$  for any  $a$ . We can choose  $a = 1$ .

For the complex conjugate pair we can use either eigenvalue. If we choose to use  $i$ , then we row reduce  $A - iI$  to get  $\begin{pmatrix} 1 & 0 & 1/2 - i/2 \\ 0 & 1 & i/2 \\ 0 & 0 & 0 \end{pmatrix}$ . If we choose the last entry of the eigenvalue to be  $2$ , the eigenvector is  $\begin{pmatrix} -1 + i \\ -i \\ 2 \end{pmatrix}$ .

The general solution is

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + C_2 \operatorname{Re} \left( \begin{pmatrix} -1 + i \\ -i \\ 2 \end{pmatrix} (\cos(t) + i \sin(t)) \right) + C_3 \operatorname{Im} \left( \begin{pmatrix} -1 + i \\ -i \\ 2 \end{pmatrix} (\cos(t) + i \sin(t)) \right) \\ &= C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -\cos(t) - \sin(t) \\ \sin(t) \\ 2 \cos(t) \end{pmatrix} + C_3 \begin{pmatrix} \cos(t) - \sin(t) \\ -\cos(t) \\ 2 \sin(t) \end{pmatrix}. \end{aligned}$$

Now we can use the initial conditions; evaluating at  $t = 0$  gives  $-C_2 + C_3 = 0$ ,  $C_1 - C_3 = 0$ , and  $2C_2 = 2$ . So  $C_3 = 1$  and  $C_1 = 1$ . For the steady state solution we drop the first term since  $e^{-t}$  will decay to 0. So the steady state solution is:

$$\begin{pmatrix} -2 \sin(t) \\ -\cos(t) + \sin(t) \\ 2 \cos(t) + 2 \sin(t) \end{pmatrix}$$

- (12) Find the equilibria of the system  $x' = 2y^3 - 2x$ ,  $y' = x^2 - 1$ , and determine their stability by computing the eigenvalues of the linearized systems.

Solution: To find the equilibria we solve the pair of equations  $2y^3 - 2x = 0$ ,  $x^2 - 1 = 0$ . The second equation is simpler, since it only involves  $x$  - any equilibria must have  $x = \pm 1$ . Substituting these values into the first equation gives  $y^3 = \pm 1$ , so  $y = \pm 1$  and  $y$  is the same sign as  $x$ . I.e. the two equilibria are  $(1, 1)$  and  $(-1, -1)$ .

The Jacobian matrix of the functions  $f_1(x, y) = 2y^3 - 2x$  and  $f_2 = x^2 - 1$  is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial(2y^3-2x)}{\partial x} & \frac{\partial(2y^3-2x)}{\partial y} \\ \frac{\partial(x^2-1)}{\partial x} & \frac{\partial(x^2-1)}{\partial y} \end{pmatrix} = \begin{pmatrix} -2 & 6y^2 \\ 2x & 0 \end{pmatrix}$$

At the equilibrium  $(1, 1)$  this becomes  $\begin{pmatrix} -2 & 6 \\ 2 & 0 \end{pmatrix}$ . The eigenvalues are solutions of  $\lambda^2 + 2\lambda - 12 = 0$ , which are  $-1 \pm \sqrt{13}$ . Since  $-1 + \sqrt{13} > 0$ , there is a positive eigenvalue and the equilibrium is unstable.

At the equilibrium  $(-1, -1)$  the Jacobian becomes  $\begin{pmatrix} -2 & 6 \\ -2 & 0 \end{pmatrix}$ . The eigenvalues are solutions of  $\lambda^2 + 2\lambda + 12 = 0$ , which are  $-1 \pm \sqrt{13}i$ . Since the real parts of these are negative, the equilibrium is stable (nearby solutions would spiral inwards).

(13) Indicate whether each of the following statements is true or false.

- (a) The set of solutions  $(x, y, z) \in \mathbb{R}^3$  to the equation  $x + y + z = 0$  is a vector subspace of  $\mathbb{R}^3$  of dimension 2.

Solution: True. A single linear homogeneous constraint will have a solution set that is one dimension less than the ambient vector space. Alternatively we can compute this by row-reducing the coefficient matrix of the system, which in this case is the matrix  $[1, 1, 1]$ . This is already in row-reduced echelon form, with one pivot and two free variables ( $y$  and  $z$ ). The number of free variables is the dimension of the solution set.

- (b) The set of solutions  $(x, y, z) \in \mathbb{R}^3$  to the equation  $x + y = 1$  is a vector subspace of  $\mathbb{R}^3$  of dimension 2.

Solution: False. This is a nonhomogeneous system, so the solutions do not form a vector subspace.

- (c) The set of solutions to the differential equation  $y'' + xy' + x^2y = 0$  is a vector space of dimension 2.

Solution: True. See Theorem 4 of section 5.2.

- (d) The set of solutions  $(x, y, z) \in \mathbb{R}^3$  of the system below is a vector subspace of  $\mathbb{R}^3$  of dimension 1.

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 9z &= 0 \end{aligned}$$

Solution: True. The coefficient matrix has a row-reduced form with two pivots and one free variable.

- (e) The polynomials  $1 + x$ ,  $1 - x$ ,  $1 + x^2$  are a basis for the vector space of polynomials with real coefficients of degree less than or equal to 2.

Solution: True. A more obvious basis would be  $1, x, x^2$ , which can be obtained from these polynomials as linear combinations:  $1 = (1 + x)/2 + (1 - x)/2$ ,  $x = (1 + x)/2 - (1 - x)/2$ , and  $x^2 = -(1 + x)/2 - (1 - x)/2 + (1 + x^2)$ .

- (14) Three identical, well-stirred tanks of with 100 liters of water in each tank are connected in series with tank 1 pumping 10 liter/minute into tank 2, tank 2 pumping 10 liter/minute into tank 3, and tank 3 pumping 10 liter/minute into tank 1. If tank 1 initially has 500 grams of salt dissolved in it, and the other two tanks start at time  $t = 0$  with no salt, which of the following initial value problems describes the amounts of salt in grams in each tank ( $x_1 =$  salt in tank 1,  $x_2 =$  salt in tank 2,  $x_3 =$  salt in tank 3).

Solution: Answer (c) is correct

$$x_1' = \frac{1}{10}x_3 - \frac{1}{10}x_1 \quad x_2' = \frac{1}{10}x_1 - \frac{1}{10}x_2 \quad x_3' = \frac{1}{10}x_2 - \frac{1}{10}x_3$$