Math 3280 Practice Midterm 2 Solutions

Please let me know if you find a typo in these solutions.

(1) Express $w = (7, -6, 14, 0)$ as a linear combination of $v_1 = (2, 3, 4, 0)$ and $v_2 = (-1, 4, -2, 0)$ or show that it is impossible to do so.

Solution: We are looking for numbers c_1 and c_2 such that $c_1 v_1 + c_2 v_2 = w$. This can be written as the matrix equation

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 4 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 14 \\ 0 \end{pmatrix}$$

The augmented coefficient matrix for this system is

$$\begin{pmatrix} 2 & -1 & 7 \\ 3 & 4 & -6 \\ 4 & -2 & 14 \\ 0 & 0 & 0 \end{pmatrix}$$

which can be row-reduced to

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{pmatrix}$$

(zero rows have been removed).

So it is possible, with $c_1 = 2$ and $c_2 = -3$.

(2) Find the general solution to the ODE: $y^{(3)} - 5y'' + 12y' - 8y = 0$.

Solution: The characteristic equation is $r^3 - 5r^2 + 12r - 8$. If we believe in a benevolent testwriter, it is natural to look for integer solutions to polynomials of degree larger than two. So we could try $1, -1, 2, -2, 4, -4, 8, -8$. Happily it is easy to check that 1 is a root, so the characteristic polynomial has $(r - 1)$ as a factor. After dividing out this factor (you should know how to do polynomial division!) we get $r^2 - 4r + 8$. From the quadratic equation we can then find the full factorization $(r-1)(r-(2-2i))(r-(2+2i))$. The general solution is $y = C_1 e^x + e^{2x}(C_2 \sin(2x) + C_3 \cos(2x))$

(3) Find the solution to the initial value problem $y'' - 2y' + 5y = e^{2x}$, $y'(0) = 0$, $y(0) = -1$.

Solution: We begin by finding the general solution $y = y_h + y_p$. The homogeneous solution y_h is determined by the characteristic equation $r^2 - 2r + 5 = (r-(1+2i))(r-(1-2i))$: $y_h = e^x(C_1 \cos(2x) + C_2 \sin(2x))$.

We can find the particular solution y_p by the method of undetermined coefficients, i.e. we suppose that $y_p = Ae^{2x}$ and solve for A. Plugging in this form and dividing out the e^{2x} factors we find that $4A - 4A + 5A = 1$, or $A = 1/5$.

Now the initial conditions can be used to determine C_1 and C_2. The condition $y(0) = -1$ becomes $C_1 + \frac{1}{5} = -1$ and $y'(0) = 0$ becomes

$$2e^x C_2 \cos(2x) + e^x C_1 \cos(2x) + e^x C_2 \sin(2x) - 2e^x C_1 \sin(2x) + \frac{2e^{2x}}{5}|_{x=0}$$
$$= \frac{2}{5} + C_1 + 2C_2 = 0.$$

The first equation can be immediately solved for $C_1 = -\frac{6}{5}$ and then the second for $C_2 = \frac{2}{5}$. So the solution is $y = e^x(-\frac{6}{5}\cos(2x) + \frac{2}{5}\sin(2x)) + \frac{1}{5}e^{2x}$.

(4) Write down the form of a particular solution y_p of the ODE $y'' + y = x^2 e^x + \cos(x)$. You do not have to determine the coefficients of the functions.

Solution: The problem is a little harder than it might look because one of the functions on the righthand side also appears in the homogeneous solution $y_h = C_1 \cos x + C_2 \sin x$. So we have to add a power of x in the undetermined particular solution: $y_p = Ax\cos x + Bx\sin x + Ce^x + Dxe^x + Ex^2e^x$.

(5) If an $n \times n$ matrix A has the property that $A^3 = 2A$, what are the possible values of the determinant of A?

Solution: Taking the determinant of both sides of the equation gives us $det(A^3) = det(2A)$. Because of the multiplicative property of determinants, $det(A^3) = (det(A))^3$. Since each row of $2A$ has been multiplied by 2, $det(2A) = 2^n det(A)$. Then we have

$$(det(A))^3 - 2^n det(A) = det(A)((det(A))^2 - 2^n) = 0$$

so either $det(A) = 0$ or $det(A) = \pm 2^{n/2}$.

(6) Find a basis for the subspace defined by the following equations for $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$:

$$-3x_1 - 3x_2 + 2x_3 - 2x_4 = 0$$
$$x_1 - 3x_2 - 4x_3 = 0$$
$$7x_1 + 15x_2 + 2x_3 + 6x_4 = 0$$

Solution: We need to row-reduce the coefficient matrix:

$$\begin{pmatrix} -3 & -3 & 2 & -2 \\ 1 & -3 & -4 & 0 \\ 7 & 15 & 2 & 6 \end{pmatrix} \xrightarrow{Swap\ R_1,R_2} \begin{pmatrix} 1 & -3 & -4 & 0 \\ -3 & -3 & 2 & -2 \\ 7 & 15 & 2 & 6 \end{pmatrix}$$

$$\xrightarrow{3R_1+R_2,-7R_1+R_3} \begin{pmatrix} 1 & -3 & -4 & 0 \\ 0 & -12 & -10 & -2 \\ 0 & 36 & 30 & 6 \end{pmatrix} \xrightarrow{3R_2+R_3} \begin{pmatrix} 1 & -3 & -4 & 0 \\ 0 & -12 & -10 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2/(-12)} \begin{pmatrix} 1 & -3 & -4 & 0 \\ 0 & 1 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{3R_2+R_1} \begin{pmatrix} 1 & 0 & -3/2 & 1/2 \\ 0 & 1 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This tells us that the first two variables are basic, or pivot, variables, while x_3 and x_4 are free variables. The number of free variables = the dimension of the solution space = the number of elements in any basis. We can read a basis off the coefficients of the free variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_3/2 - x_4/2 \\ -5x_3/6 - x_4/6 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 3/2 \\ -5/6 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1/2 \\ -1/6 \\ 0 \\ 1 \end{pmatrix}$$

I.e., one basis is $\left\{ \begin{pmatrix} 3/2 \\ -5/6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/6 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(7) Solve the initial value problem $y''' - 27y = e^{3x}$, $y(0) = y'(0) = y''(0) = 0$.

Solution:

First we find the homogeneous (also called complementary) solution to

$$y_c''' - 27y_c = 0.$$

To do this we have to factor the characteristic equation $r^3 - 27 = 0$.

One root is easy to get: $r_1 = (27)^{1/3} = 3$.

If we divide $r^3 - 27$ by $r - 3$, the quotient is $r^2 + 3r + 9$.

With the quadratic formula we can get the other two roots, $r_2, r_3 = -\frac{3}{2} \pm \frac{3\sqrt{3}i}{2}$. With these three roots, we can construct the complementary solution:

$$y_c = C_1 e^{3x} + C_2 e^{-\frac{3t}{2}} \cos(\frac{3\sqrt{3}t}{2}) + C_3 e^{-\frac{3t}{2}} \sin(\frac{3\sqrt{3}t}{2})$$

Next, to find the particular solution we would normally use the method of undetermined coefficients with the form $y_p = Ae^{3x}$.

But this is contained within the complementary solution, so instead we use

$$y_p = Axe^{3x}.$$

Since $y_p''' = 27xAe^{3x} + 27Ae^{3x}$, we require that

$$\begin{aligned} y_p''' - 27y_p &= 27xAe^{3x} + 27Ae^{3x} - 27xAe^{3x} \\ &= 27Ae^{3x} = e^{3x} \end{aligned}$$

and so $A = 1/27$.

So the general solution to the ODE is

$$y = y_c + y_p = C_1 e^{3x} + C_2 e^{-\frac{3t}{2}} \cos\left(\frac{3\sqrt{3}t}{2}\right) + C_3 e^{-\frac{3t}{2}} \sin\left(\frac{3\sqrt{3}t}{2}\right) + \frac{1}{27} x e^{3x}$$

The initial condition $y(0) = 0$ becomes $C_1 + C_2 = 0$. Since

$$y' = -\frac{3}{2} (\sqrt{3}C_2 + C_3) e^{-\frac{3x}{2}} \sin\left(\frac{3\sqrt{3}}{2}x\right) + \frac{3}{2} (\sqrt{3}C_3 - C_2) e^{-\frac{3x}{2}} \cos\left(\frac{3\sqrt{3}}{2}x\right) + (3\,C_1 + \frac{1}{27} + \frac{x}{9}) e^{3x}$$

$$y'(0) = \frac{3}{2}\sqrt{3}C_3 - \frac{3}{2}\,C_2 + 3\,C_1 + \frac{1}{27} == 0$$

Now we compute the equation for the initial condition $y''(0) = 0$

$$y'' = \frac{9}{2} e^{-\frac{3x}{2}} \left((\sqrt{3}C_2 - C_3) \sin\left(\frac{3\sqrt{3}}{2}x\right) + (\sqrt{3}C_3 + C_2) \cos\left(\frac{3\sqrt{3}}{2}x\right) \right) + e^{3x} \left(9C_1 + \frac{2}{9} + \frac{x}{3} \right)$$

$$y''(0) = -\frac{9}{2}\sqrt{3}C_3 - \frac{9}{2}\,C_2 + 9\,C_1 + \frac{2}{9} = 0$$

Writing all of these initial conditions as a matrix-vector system we get:

$$\begin{pmatrix} 1 & 1 & 0 \\ 3 & -\frac{3}{2} & \frac{3}{2}\sqrt{3} \\ 9 & -\frac{9}{2} & -\frac{9}{2}\sqrt{3} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/27 \\ -2/9 \end{pmatrix}$$

The row-reduced echelon form of the augmented coefficient matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{81} \\ 0 & 1 & 0 & \frac{1}{81} \\ 0 & 0 & 1 & \frac{1}{243}\sqrt{3} \end{pmatrix}$$

So finally we have:

$$y = \frac{1}{81} \left[e^{-\frac{3x}{2}} \left(\frac{\sqrt{3}}{3} \sin\left(\frac{3\sqrt{3}}{2}x\right) + \cos\left(\frac{3\sqrt{3}}{2}x\right) \right) + (3x - 1)\, e^{3x} \right]$$