Math 3280 Practice Final Solutions

This is somewhat longer than the actual exam, which will probably be 6 questions. Please let me know if you find any typos in these solutions.

(1) Find the general solution to $(1+t)y' + y = \cos t$.

Solution: In standard form ($y' + P(t)y = Q(t)$) we have $y' + \frac{1}{1+t}y' = \frac{\cos(t)}{1+t}$. Using the integrating factor method, we have $\rho(t) = e^{\int P(t)dt} = e^{\log(1+t)} = 1+t$. Then $\int \rho\, Q\, dt = \int \cos t\, dt = \sin t$ and $y = \frac{C}{1+t} + \frac{\sin t}{1+t}$.

(2) Rewrite the initial value problem $y''' + y'' + y = t$, $y(0) = y'(0) = y''(0) = 0$ as an equivalent first-order system.

Solution: Introduce the variables $v_1 = y'$, $v_2 = v_1' = y''$ and the system becomes:

$$y' = v_1$$

$$v_1' = v_2$$

$$v_2' = t - v_2 - y$$

$$y(0) = 0,\ v_1(0) = 0,\ v_2(0) = 0$$

(3) Find the general solution to the system

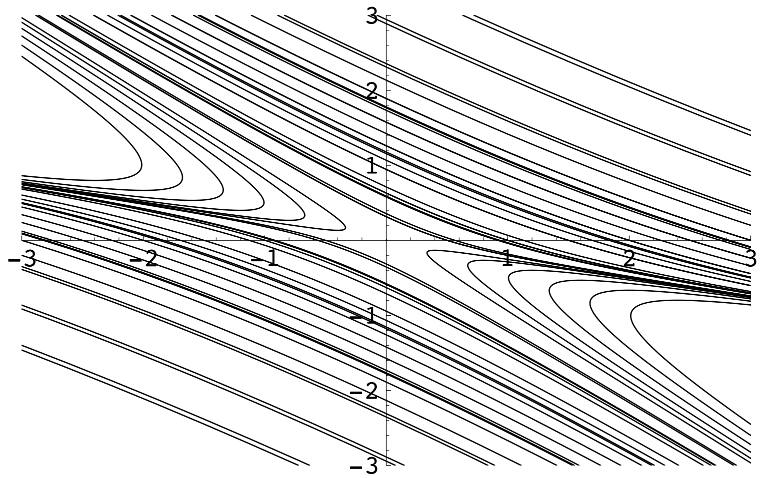
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The eigenvalues of the matrix are found from

$$det\begin{bmatrix} 2-\lambda & 4 \\ -1 & -3-\lambda \end{bmatrix} = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$$

From row-reducing $A - \lambda I$ we can find that the eigenvectors are $\vec{v_1} = (-4, 1)$ and $\vec{v_2} = (-1, 1)$, so the solutions are $x_1 = -4C_1e^t - C_2e^{-2t}$ and $x_2 = C_1e^t + C_2e^{-2t}$.

For large t, $(x_1, x_2) \approx e^t(-4C_1, C_1)$. For large $-t$, $(x_1, x_2) \approx e^{-2t}(-C_2, C_2)$. Some trajectories are shown below.



(4) Are the vectors $v_1 = (1, 2, 3, 4)$, $v_2 = (2, -2, 4, 2)$, and $v_3 = (0, -3, -1, -3)$ linearly independent? If not, write one of them as a linear combination of the other two.

Solution: The vectors are linearly dependent if there are c_1, c_2, c_3, not all zero, such that $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. This is equivalent to the coefficient matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & -3 \\ 3 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix}$ having less than 3 pivots after row-reduction. If we row-reduce A we find

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & -3 \\ 3 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -6 & -3 \\ 0 & -2 & -1 \\ 0 & -6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This only has two pivots. The conditions on the c_i are $c_1 + 2c_2 = 0$ and $2c_2 + c_3 = 0$. If we choose $c_2 = 1$ then we have $(c_1, c_2, c_3) = (-2, 1, -2)$, or $v_2 = 2v_1 + 2v_3$.

(5) Solve the initial value problem $y'' + y = \cos x$, $y'(0) = 0$, $y(0) = -\frac{1}{2}$.

Solution: This could be also done with a Laplace transform. Using undetermined coefficients we find the solution by decomposing it into $y = y_h + y_p$. The homogeneous solution y_h is found from the characteristic equation $r^2 + 1 = (r - i)(r + i) = 0$ to be $y_h = C_1 \cos(x) + C_2 \sin(x)$.

Since the right-hand side $\cos(x)$ is contained in the solution space of the homogeneous equation, we are forced to consider particular solutions of the form $y_p = Ax\cos(x)+Bx\sin(x)$. Then $y_p'' = -Ax\cos(x)-2A\sin(x)+2B\cos(x)-Bx\sin(x)$. Substituting these forms into our ODE yields $-2A\sin(x)+2B\cos(x) = \cos(x)$, so $A = 0$ and $B = 1/2$.

So now we know that $y = C_1\cos(x) + C_2\sin(x) + x\sin(x)/2$. The initial conditions become $C_2 = 0$ and $C_1 = -\frac{1}{2}$, so $y = \frac{-\cos(x)+x\sin(x)}{2}$.

(6) Use Euler's, the Improved Euler's, or the Runge-Kutta method to numerically approximate $y(2)$ to three digits of accuracy if $y' = t + \sqrt{y}$ and $y(0) = 1$.

Solution: It takes 715 steps to get the desired accuracy with Euler's Method. For the improved Euler's method, 14 steps are needed. Fourth-order Runge-Kutta works in 2 steps (stepsize 1), giving $y(2) \approx 6.407$ which agrees with $y(2) = 6.41147412780977283851 3\ldots$ in the first three digits after rounding.

(7) Find the general solution to the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The characteristic equation is $\det(A - \lambda I) = \lambda^2 - 4\lambda + 8$ with roots (eigenvalues) $\lambda = 2 \pm 2i$. We need to find one eigenvector, lets find it for $\lambda = 2 + 2i$. So we row reduce

$$A - (2+2i)I = \begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \to \begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}$$

So the eigenvector can be chosen to be $v = (-1+2i, 1)$. Then the solution to the system is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 Re[ve^{2t}(\cos(2t) + i\sin(2t))] + C_2 Im[ve^{2t}(\cos(2t) + i\sin(2t))]$$

$$= C_1\begin{bmatrix} -e^{2t}(\cos(2t) + 2\sin(2t)) \\ e^{2t}\cos(2t) \end{bmatrix} + C_2\begin{bmatrix} e^{2t}(-\sin(2t) + 2\cos(2t)) \\ e^{2t}\sin(2t) \end{bmatrix}$$

(8) Find the Laplace transform $X(s) = \mathcal{L}(x(t))$ if $x'' + 8x' + 15x = 0$ and $x(0) = 0$, $x'(0) = 1$. Then find $x(t)$ using any method.

Solution: Taking the Laplace transform of the ODE gives

$$s^2X(s) + 8sX(s) + 15X(s) - 8x(0) - sx(0) - x'(0)$$

$$= s^2X(s) + 8sX(s) + 15X(s) - 1 = 0.$$

Solving for $X(s)$ and performing a partial fraction decomposition, we get

$$X(s) = \frac{1}{s^2 + 8s + 15} = \frac{1/2}{s+3} - \frac{1/2}{s+5}$$

Since $\mathcal{L}^{-1}(\frac{1}{s-a}) = e^{at}$, we can invert $X(s)$ to get $x(t) = \frac{e^{-3t}}{2} - \frac{e^{-5t}}{2}$.

(9) What is the **form** of the general solution to the ODE $y''' - 4y'' + 14y' - 20y = te^t \cos(3t) + t^2$. Hint: one of the roots of the characteristic polynomial of the left-hand side is 2.

Solution:

The homogeneous solutions is $y_h = C_1e^t \sin(3t) + C_2e^t \cos(3t) + C_3e^{2t}$. The form of the particular solution is $y_p = At^2e^t \cos(3t) + Bt^2e^t \sin(3t) + Cte^t \cos(3t) + Dte^t \sin(3t) + Et^2 + Ft + G$. The form of the general solution is the sum of these, $y = y_h + y_p$.