Math 3280 Assignment 6, due Thursday, July 25th.

Find the eigenvalues and eigenvectors of the following matrices:

(1) $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

(2) $\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$

(3) $\begin{pmatrix} 2 & 0 & 0 \\ 5 & 3 & -2 \\ 2 & 0 & 1 \end{pmatrix}$

(4) $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(5) $\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$

(6) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Find a matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix, for the following matrices if such a P exists.

(7) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

(8) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

(9) Show that if A is invertible and λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}. Are the eigenvectors the same?

(10) By computing the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ find a matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. Use this diagonalization to compute A^6.

(11) Find the general solution to the system $x_1' = x_1 + 2x_2$, $x_2' = 2x_1 + x_2$. Sketch some of the solutions near the origin, including some that start on the lines spanned by the eigenvectors of the coefficient matrix of the system.

(12) Find the general solution to the system $x_1' = x_1 + 2x_2$, $x_2' = 3x_1 + 2x_2$.

(13) Find the general solution to the system $x_1' = x_1 - 5x_2$, $x_2' = x_1 - x_2$. Sketch some of the solutions near the origin.

(14) Solve the initial value problem $x_1' = x_1 + 2x_2$, $x_2' = -2x_1 + x_2$, $x_1(0) = 1$, $x_2(0) = 0$.

(15) Find the error between the exact values of $x_1(1)$ and $x_2(1)$ and an approximation using Euler's method for the initial value problem $x_1' = 9x_1 + 5x_2$, $x_2' = -6x_1 - 2x_2$, $x_1(0) = 0$, $x_2(0) = 1$.

(16) Suppose two 50 liter tanks are connected by two pumps which transfer 10 liters/minute of fluid from each tank to the other. Suppose that the first tank initially contains 50 liters of brine at a concentration of 0.2 kg of salt per liter, and the other tank contains 50 liters of pure water.

(a) Find the amount of salt in each tank as a function of time (you can assume that the tanks are well-stirred).

(b) How long will it take for the amount of salt in the second tank to be within 1% of the amount of salt in the first tank?

(17) Find the general solution of $x' = Ax$ if

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

For the next three problems, consider two blocks of mass m_1 and m_2 connected by springs to each other and to walls as shown below. The displacement of the masses from their equilibrium positions are denoted by x_1 and x_2. The stiffness of the three springs are k_1, k_2, and k_3 as shown. Compute the natural frequencies and describe the natural modes of oscillation in each of the three following cases:

(18) $k_1 = k_2 = 4$ and $k_3 = 2$, and $m_1 = 2$, $m_2 = 1$.

(19) $k_1 = k_3 = 1$ and $k_2 = 4$, and $m_1 = m_2 = 1$.

(20) $k_1 = k_3 = 0$ and $k_2 = 4$, and $m_1 = m_2 = 1$.