Math 3298 Practice Final Solutions

Please let me know if you find any mistakes or typos.

(1) Reverse the order of integration for the integral $\int_0^1 \int_x^1 \int_0^{y^2} f(x,y,z)dzdydx$.

Solution: The answer is:
$$\int_0^1 \int_{\sqrt{z}}^1 \int_0^y f(x,y,z)\ dx\ dy\ dz.$$
The projection onto the xy plane is the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$. The surface slopes up in the z-direction parabolically to the line $y=1$.

(2) Compute the vector line integral $\int_C \vec{F}\cdot d\vec{r}$ where C is the path $(4-3^t, -2+2t, \pi t)$, $t \in [0,1]$, and $\vec{F} = (2x\cos z - x^2, z-2y, y-x^2\sin z)$.

Solution: The field is conservative since $curl(\vec{F}) = \vec{0}$. A potential function for $\vec{F}$ is $f = x^2\cos z - x^3/3 + yz - y^2$, so $\int_C \vec{F}\cdot d\vec{r} = f(1,0,\pi) - f(3,-2,0) = -4/3 - (9-9-4) = 8/3$.

(3) Find the linearization of $f(x,y)$ at $(x,y) = (0,1)$ if $f = h(u(x,y), v(x,y))$ and $\text{grad}(h)|_{(1,1)} = (\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v})|_{(1,1)} = (2,3)$, $u(x,y) = x+y$, and $v(x,y) = y^2$.

Solution: The linearization is $L(x,y) = f(0,1) + \frac{\partial f}{\partial x}|_{(0,1)}(x-0) + \frac{\partial f}{\partial y}|_{(0,1)}(y-1)$. To find the partial derivatives we use the chain rule and the derivatives $\frac{\partial u}{\partial x}| = 1$, $\frac{\partial u}{\partial y} = 1$, $\frac{\partial v}{\partial x}| = 0$, $\frac{\partial v}{\partial y}|_{0,1} = 2$:
$$\frac{\partial f}{\partial x}|_{(0,1)} = \frac{\partial h}{\partial u}|_{(1,1)}\frac{\partial u}{\partial x}|_{(0,1)} + \frac{\partial h}{\partial v}|_{(1,1)}\frac{\partial v}{\partial x}|_{(0,1)} = 2\cdot 1 + 3\cdot 0 = 2$$
$$\frac{\partial f}{\partial y}|_{(0,1)} = \frac{\partial h}{\partial u}|_{(1,1)}\frac{\partial u}{\partial y}|_{(0,1)} + \frac{\partial h}{\partial v}|_{(1,1)}\frac{\partial v}{\partial y}|_{(0,1)} = 2\cdot 1 + 3\cdot 2 = 8$$
So
$$L(x,y) = f(0,1) + 2(x-0) + 8(y-1) = f(0,1) + 2x + 8y - 8.$$
Since neither f nor h is given explicitly this is all that is possible to determine, besides the fact that $f(0,1) = h(1,1)$.

(4) Find the surface area of the torus parameterized by $x = (2+\cos(v))\cos(u)$, $y = (2+\cos(v))\sin(u)$, $z = \sin(v)$, with $u \in [0,2\pi]$ and $v \in [0,2\pi]$.

Solution: The surface area element is computed from the length of the cross product of the partials $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$, where $\vec{r} = (x, y, z)$. After many uses of the identity $\sin^2(t) + \cos^2(t) = 1$, this simplifies to

$$dS = |\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}| dudv = |2 + \cos(v)| dudv.$$

The absolute value function can be dropped because $2 + \cos(v) > 0$. Now the surface area can be computed:

$$S.A. = \int_0^{2\pi} \int_0^{2\pi} (2 + \cos(v)) dudv = 2\pi \int_0^{2\pi} (2 + \cos(v)) dv = 8\pi^2.$$

(5) Find the maxima and minima of $f(x, y) = \frac{1}{x} + \frac{2}{y}$ on the set $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

Solution: I will use the Lagrange multiplier method. Let $g = \frac{1}{x^2} + \frac{1}{y^2} - 1$ (the constraint), and then require that $\bigtriangledown(f) = \lambda \bigtriangledown (g)$, i.e.

$$(-\frac{1}{x^2}, -\frac{2}{y^2}) = (-\frac{2\lambda}{x^3}, -\frac{2\lambda}{y^3}).$$

After clearing denominators we find that $\lambda = x/2 = y$. Using that relation between x and y, the constraint equations becomes $g(x, x/2) = \frac{1}{x^2} + \frac{4}{x^2} - 1 = 0$ or $x^2 = 5$. So there are two critical points on the constraint curve, $\pm(\sqrt{5}, \sqrt{5}/2)$. Comparison with other values of f on the constraint curve shows that $f(\sqrt{5}, \sqrt{5}/2) = \sqrt{5}$ is a maximum and $f(-\sqrt{5}, -\sqrt{5}/2) = -\sqrt{5}$ is a minimum.

(6) Find the volume of the solid wedge bounded by the planes $z = 0$ and $z = -2y$ and the cylinder $x^2 + y^2 = 4$ (with $y \geq 0$).

Solution: In cylindrical coordinates

$$V = \int_0^{\pi} \int_0^{2} \int_{-2r\sin\theta}^{0} dz\ r\ dr\ d\theta = \int_0^{\pi} \int_0^{2} 2r^2 \sin\theta\ dr\ d\theta$$

$$= \int_0^{\pi} 16 \sin\theta/3\ d\theta = 32/3.$$

(7) Use Green's Theorem to find the smooth, simple, closed and positively oriented curve in the plane for which the line integral $\oint (\frac{x^2 y}{4} + \frac{y^3}{3}) dx + x dy$ has the largest possible value.

Solution: The corresponding double integral from Green's theorem is

$$\int\int_R (1 - x^2/4 - y^2)\ dA.$$

The integrand is positive in the interior of the ellipse $x^2/4+y^2 = 1$, so we choose this ellipse as the desired curve ($x = 2\cos(t)$, $y = \sin t$).

(8) Compute the value of $\int\int_S (\nabla \times \vec{F}) \cdot \vec{n}\ dS$ where S is the upper half of the ellipsoid $4x^2 + 9y^2 + 36z^2 = 36$, $z \geq 0$, with upward pointing normal, and $\vec{F} = (y, x^2, (x^2 + y^2)^{3/2}e^{xyz})$.

Solution: The presence of nasty stuff like e^{xyz} inspires us to reformulate the computation using Stokes' theorem. The ellipse boundary can be parameterized as $x = 3\cos t$, $y = 2\sin t, z = 0$. The corresponding line integral is then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (2\sin t, 9\cos^2 t, (9\cos^2(t) + 4\sin^2(t))^{3/2}) \cdot (-3\sin t, 2\cos t, 0)dt =$$

$$\int_0^{2\pi} -6\sin^2(t) + 18\cos^3(t)\ dt = -6\pi.$$

(9) Let $\vec{r}(t)$ be a curve in space with unit tangent, normal, and binormal vectors $\vec{T}$, $\vec{N}$, and $\vec{B}$. Show that $\frac{d\vec{B}}{dt}$ is perpendicular to $\vec{T}$.

Solution: Since $\vec{B} = \vec{T} \times \vec{N}$, $\frac{d\vec{B}}{dt} = \frac{d\vec{T}}{dt} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{dt}$. However $\vec{N}$ is parallel to $\frac{d\vec{T}}{dt}$, so in fact $\frac{d\vec{B}}{dt} = \vec{T} \times \frac{d\vec{N}}{dt}$. Finally, note that $\vec{a}$ is always perpendicular to $\vec{a} \times \vec{b}$ for any vectors $\vec{a}$, $\vec{b}$.

(10) Compute the flux integral $\int\int_S \vec{F} \cdot \vec{n}\ dS$ where S is the graph of $z = 1 - x^2 - y^2$, with upward normal, for $z \geq 0$, and with $\vec{F} = (xz, yz, 2z^2)$.

Solution: The given flux integral can be computed directly or by using the divergence theorem. I will do it both ways for comparison (on a test, one method would be sufficient). Directly: since the surface is given as a graph $z = f(x, y)$, $\vec{n}\ dS = d\vec{S} = (-f_x, -f_y, 1) = (2x, 2y, 1)$. Then

$$\int\int_S \vec{F} \cdot \vec{n}\ dS = \int\int_S (xz, yz, 2z^2)|_S \cdot (2x, 2y, 1)\ dxdy$$

$$= \int\int_S (1-x^2-y^2)(2x^2+2y^2)+2(1-x^2-y^2)^2\ dxdy = 2\int_0^{2\pi}\int_0^1 (1-r^2)(2r^2+2(1-r^2))r\ drd\theta$$
$$= 4\int_0^{2\pi}\int_0^1 (r-r^3)\ drd\theta = \pi.$$

We cannot immediately apply the divergence theorem because this surface is not closed. However, we could consider the closed surface $S_2 = S \cup S_1$, where S_1 is the unit disk $z = 0$, $x^2 + y^2 \leq 1$ and with normal $(0, 0, -1)$. Since $\vec{F} = 0$ on S_1, this addition doesn't actually affect the flux integral, i.e. $\int\int_{S_2} \vec{F} \cdot \vec{n}\ dS_2 = \int\int_S \vec{F} \cdot \vec{n}\ dS$ but $\int\int_{S_2} \vec{F} \cdot \vec{n}\ dS_2 = \int\int\int div\vec{F} dV$. Since $div\vec{F} = 6z$, the value we are after can be computed in cylindrical coordinates as:

$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6zr\ dzdrd\theta = 3\int_0^{2\pi}\int_0^1\int_0^{1-r^2} (1-r^2)^2 r\ drd\theta = \pi.$$

(11) Use the divergence theorem to compute the flux of $\vec{F} = (z^5 + x, \cos(xz), z^2)$ through the surface bounded by $z = 0$ and $z = 1 - x^2 - y^2$.

Solution: The divergence of $\vec{F}$ is $1 + 2z$, so

$$\int\int_S \nabla \times \vec{F} \cdot d\vec{S} = \int\int\int_R (1+2z)dV$$

where R is the interior of S. To evaluate this it is easiest to use cylindrical coordinates:

$$\int_0^1\int_0^{2\pi}\int_0^{1-r^2} (1+2z)r\ dz\ d\theta\ dr = \int_0^1\int_0^{2\pi} (2r-3r^3+r^5)\ d\theta\ dr = 5/12\int_0^{2\pi} d\theta = 5\pi/6$$