Math 3298 Practice Midterm 1 Solutions

Please notify me as soon as possible if you believe there is an error in these solutions.

(1) (a) Use the formula $\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$ to show that for a parameterized plane curve $(x(t), y(t))$ the curvature is

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{|\dot{x}^2 + \dot{y}^2|^{3/2}}$$

Solution: In order to use the three-dimensional curvature formula we can embed the planar curve $(x(t), y(t))$ into space as $\vec{r}(t) = (x(t), y(t), 0)$. Then only the third component of the cross-product $\vec{r}\,' \times \vec{r}\,''$ is nonzero (since the first and second derivative are in the x-y plane and the cross-product must be perpendicular to them), with magnitude $|\dot{x}\ddot{y} - \ddot{x}\dot{y}|$.

(b) Use the result of part (a) to compute the curvature of $x(t) = 1 + t^3$, $y(t) = t + t^2$.

Solution: We compute all the pieces: $x' = 3t^2$, $x'' = 6t$, $y' = 1 + 2t$, $y'' = 2$. Then the curvature is

$$\kappa = \frac{|6(t+1)t|}{(9\,t^4 + 4\,t^2 + 4\,t + 1)^{\frac{3}{2}}}$$

(2) Classify the critical points of $f(x, y) = 2y^2 + 2xy - y - x^3 + x + 1$.

Solution: To find the critical points we need to solve the system

$$\frac{\partial f}{\partial x} = -3x^2 + 2y + 1 = 0$$

$$\frac{\partial f}{\partial y} = 4y + 2x - 1 = 0$$

There are many ways to proceed. One way is to eliminate y by subtracting two times the first equation from the second:

$$\frac{\partial f}{\partial y} - 2\frac{\partial f}{\partial x} = 6x^2 + 2x - 3 = 0.$$

This has roots $x = -1/6 \pm \sqrt{19}/6$, and corresponding y-values $y = 1/3 \mp \sqrt{19}/12$. To classify these two critical points we compute the discriminant

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2 = (-6x)(4) - (2)^2 = -24x - 4$$

At $(-1/6 + \sqrt{19}/6, 1/3 - \sqrt{19}/12)$ we have $D < 0$ so it is a saddle point. At $(-1/6 - \sqrt{19}/6, 1/3 + \sqrt{19}/12)$, $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ so that point is a relative minimum.

(3) Compute the limit $\dfrac{x^2 + y\sin(y)}{x^2 + y^2}$ if it exists, or show why it does not exist.

Solution: One way to think about this is to expand $\sin(y)$ in a power series around $y = 0$, since the series for $\sin(y)$ converges everywhere and it will behave like its lowest order term for small $|y|$:

$$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots$$

so we would expect the function to behave like

$$\frac{x^2 + y^2}{x^2 + y^2} - \frac{y^4/6}{x^2 + y^2} + \dots = 1 - \frac{y^4/6}{x^2 + y^2} + \dots$$

and for the limit to exists and be equal to 1. A proof is not expected on this type of problem, but it is not too hard to prove if we assume the fact that $|sin(y) - y|$ is $O(y^3)$ as $y \to 0$ - i.e. as y gets close to 0, $\sin(y) = y + \epsilon(y)$, where $\epsilon(y)$ is a function such that there exists positive constant M, y_0 where $|\epsilon(y)| < My^3$ for all $|y| < |y_0|$.

Another approach would be to check the behavior along straight lines through the origin, such as $y = mx$. The limit of the function on such a line as $x \to 0$ can be computed by using L'Hopital's Rule, and it is 1 for all m. This is not a proof, but it is usually strong evidence that the limit exists.

(4) Find the curvature of $\vec{r}(t) = (t^2, t^3, 2t^3)$ at $t = 1$.

Solution: The first two derivatives are $\vec{r}$ are $\vec{r}' = (2t, 3t^2, 6t^2)$ and $\vec{r}'' = (2, 6t, 12t)$. At $t = 1$ these reduce to $(2, 3, 6)$ and $(2, 6, 12)$ respectively. Now we can compute the curvature:

$$\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{6\sqrt{5}}{343}$$

(The cross product at $t = 1$ is $\vec{r}' \times \vec{r}'' = (0, -12, 6)$.)

(5) Use the linearization of the function $f(x, y) = x + \ln xy$ at $(x, y) = (2, 1/2)$ to find an approximate value for $f(1.9, .4)$.

Solution: The linearization is

$$L(x, y) = f|_{(2,1/2)} + \frac{\partial f}{\partial x}|_{(2,1/2)}(x - 2) + \frac{\partial f}{\partial y}|_{(2,1/2)}(y - 1/2).$$

$$= 2 + \ln 1 + (1 + \frac{1}{2})(x - 2) + (2)(y - 1/2) = -2 + 3x/2 + 2y$$

since $\frac{\partial f}{\partial x} = 1 + 1/x$ and $\frac{\partial f}{\partial y} = 1/y$.

So our approximation is $L(1.9, 4) = 1.65 \approx f(1.9, .4) = 1.62556\dots$

(6) Find three positive numbers x, y, and z such that $x+2y+3z=7$ and for which the function $f(x,y,z)=x^2y^2z^3$ is maximized.

Solution: This can be done by elimination or Lagrange multipliers. The latter is slightly simpler. The solution will lie on the constraint $g=x+2y+3z-7=0$. Critical points of f restricted to the set $g=0$ will have $\lambda\ grad(g)=grad(f)$ or

$$\lambda\ (1,2,3)=(\lambda,2\lambda,3\lambda)=(2xy^2z^3,2x^2yz^3,3x^2y^2z^2)$$

Dividing the first equation by the second and third gives $1/2=y/x$ and $1/3=2z/(3x)$, so $y=x/2$ and $z=x/2$. Plugging these into $g=0$ gives $x+2(x/2)+3(x/2)-7=7x/2-7=0$ so $x=2$, $y=1$, and $z=1$.

(7) Use the chain rule to compute $\frac{\partial z}{\partial t}$ at $t=2$ if $z=\sin(xy)\sin(y)$ and $x=1/t$, $y=f(t)$ where $f'(2)=3$ and $f(2)=\pi$.

Solution:

$$\frac{\partial z}{\partial t}|_{t=2}=(\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt})|_{t=2}$$

$$=(y\cos(xy)\sin(y))(\frac{-1}{t^2})+(x\cos(xy)\sin(y)+\sin(xy)\cos(y))f'(t)|_{t=2}$$

$$=(\pi\cos(x\pi)\sin(\pi))(\frac{-1}{t^2})+(x\cos(x\pi)\sin(\pi)+\sin(x\pi)\cos(\pi))f'(t)$$

$$=(\pi\cos(\pi/2)\sin(\pi))(\frac{-1}{4})+(\frac{1}{2}\cos(\pi/2)\sin(\pi)+\sin(\pi/2)\cos(\pi))3$$

$$=-3$$

(8) Find the directions in which the directional derivative of $f(x,y)=x^2+2y^2-4y$ at the point $(1,1)$ has the value 1.

Solution: The directional derivative of f at $(1,1)$ in the direction $\vec{u}=(u_1,u_2)$ is

$$\nabla f|_{(1,1)}\cdot\vec{u}=\frac{\partial f}{\partial x}|_{(1,1)}u_1+\frac{\partial f}{\partial y}|_{(1,1)}u_2.$$

So we compute the partial derivatives of f and evaluate them at $(1,1)$: $\frac{\partial f}{\partial x}|_{(1,1)}=2x|_{(1,1)}=2$, $\frac{\partial f}{\partial y}|_{(1,1)}=(4y-4)|_{(1,1)}=0$. Then the condition on the directional derivative becomes

$$\nabla f|_{(1,1)}\cdot\vec{u}=(2,0)\cdot(u_1,u_2)=2u_1=1,$$

so $u_1=1/2$. Since $\vec{u}$ is a unit vector $u_1^2+u_2^2=1$, and so $u_2^2=3/4$ and the two possible directions are $\vec{u}=(1/2,-\sqrt{3}/2)$ and $\vec{u}=(1/2,\sqrt{3}/2)$.

(9) Find the integral of the function $f(x,y) = 2x\sqrt{y^2 - x^2}$ over the triangle $T = \{(x,y) \mid 0 \le y \le 2, 0 \le x \le y\}$

Solution: This can be done in either order but its easier to do the x-integral first:

$$\int_0^2 \int_0^y 2x\sqrt{y^2 - x^2}dxdy = -\frac{2}{3}\int_0^2 (y^2 - x^2)^{3/2}|_0^y dy$$

$$= \frac{2}{3}\int_0^2 y^3 dy = \frac{y^4}{6}|_0^2 = 8/3$$

The first integral can be done with a substitution $u = y^2 - x^2$.