Math 3298 Practice Midterm 2 Solutions

Please let me know if you think there are errors in any of the solutions.

(1) Find the average value of the function $1 + 3x + y$ on the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$.

Solution: The area of the triangle is equal to one, so the average value will simply be the integral

$$\int_0^1 \int_0^{2-2x} (1 + 3x + y)\ dy\ dx = \int_0^1 (4 - 4x^2)\ dx = \frac{8}{3}$$

(2) Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cylinder $x^2 + y^2 = 1$.

Solution: This is probably easiest in cylindrical coordinates. Solving the sphere boundary equation for z we find $z = \pm\sqrt{9 - x^2 - y^2} = \pm\sqrt{9 - r^2}$. So the volume is

$$\int_0^{2\pi} \int_1^3 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r dz dr d\theta = \int_0^{2\pi} \int_1^3 2r\sqrt{9 - r^2} dr d\theta$$

$$= \int_0^{2\pi} -\frac{2}{3}(9 - r^2)^{3/2}|_1^3 d\theta = \frac{4\pi\ 8^{3/2}}{3}$$

(3) Compute the integral $\int\int\int_R \sqrt{x^2 + y^2}\ dV$ where R is the region inside the cylinder $x^2 + y^2 = 25$ and between $z = -1$ and $z = 4$.

Solution: Again, cylindrical coordinates are the best choice for this problem.

$$\int\int\int_R \sqrt{x^2 + y^2}\ dV = \int_0^5 \int_0^{2\pi} \int_{-1}^4 r^2\ dz\ d\theta\ dr = \int_0^5 \int_0^{2\pi} 5r^2\ d\theta\ dr$$

$$= \int_0^5 10\pi r^2\ dr = \frac{10\pi r^3}{3}|_0^5 = \frac{1250\pi}{3}$$

(4) Find the volume of the solid bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$.

Solution: This is a tetrahedron. By considering any three of the four boundary equations, we can find that the vertices are $(0, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$, and

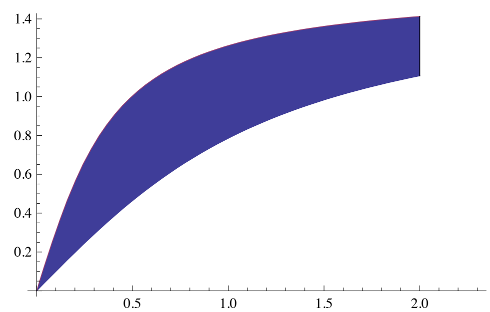
$(0,2,0)$. This helps to sketch the figure and determine the bounds for the integral:

$$V = \int_0^1 \int_x^{2-x} \int_0^x dz\ dy\ dx = \int_0^1 \int_x^{2-x} x\ dy\ dx = \int_0^1 xy|_x^{2-x}\ dx$$

$$= \int_0^1 (2x - 2x^2)\ dx = 1/3$$

(5) Change the order of integration of $\int_0^2 \int_{Arctan(x)}^{Arctan(\pi x)} dydx$ and evaluate the integral.

Solution: This question is definitely a bit harder than one I would put on an exam. The integration region is shown below.



To do the x-integral first we need to split up the region into two pieces because of the corner at $(2, \arctan(2))$. Then we have

$$\int_0^{\arctan(2)} \int_{\tan(y)/\pi}^{tan(y)} dx\ dy + \int_{\arctan(2)}^{\arctan(2\pi)} \int_{\tan(y)/\pi}^{2} dx\ dy =$$

$$\int_0^{\arctan(2)} (\tan(y) - \tan(y)/\pi) dy + \int_{\arctan(2)}^{\arctan(2\pi)} (2 - \tan(y)/\pi) dy =$$

$$-2\arctan(2) + 2\arctan(2\pi) + \frac{(-1+\pi)\log(5)}{2\pi} + \frac{\log\left(\frac{5}{1+4\pi^2}\right)}{2\pi} \approx .827\ldots$$

The final answer has been simplified using several properties of the logarithm: $\log(a) - \log(b) = \log(a/b)$, $\log(1/b) = -\log(b)$, and $\log(a^b) = b\log(a)$.

(6) Compute the integral $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{3/2}\ dz\ dy\ dx$ by changing to cylindrical coordinates.

Solution: The projection of the region onto the x-y plane is the disk of radius 1. So the integral can be rewritten in cylindrical coordinates as:

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r^4 \ dz \ dr \ d\theta$$

which evaluates to

$$= \frac{1}{5} \int_0^{2\pi} \int_0^1 (2 - 2r^2) r^4 \ dr \ d\theta = 8\pi/35$$

(7) This difficult a problem would be extra credit: Assuming that $\beta \in (0, \pi/2)$ and $a > 0$, compute the following integral

$$\int_0^{a \sin\beta} \int_{y \cot\beta}^{\sqrt{a^2-y^2}} \int_0^1 \ln{(x^2 + y^2)} \ dz \ dx \ dy$$

Solution: The z-integral is easy and we get

$$\int_0^{a \sin\beta} \int_{y \cot\beta}^{\sqrt{a^2-y^2}} \ln{(x^2 + y^2)} \ dx \ dy$$

The first thing to do is understand the region of integration. The upper x-boundary $x = \sqrt{a^2 - y^2}$ is the right-hand semicircle of radius a centered at $(0, 0)$. The lower x-boundary is the line $x = \cot{(\beta)} y$ or $y = \tan{(\beta)} x$, a line through $(0, 0)$ with angle β. The y boundaries are the x-axis and the height where the line intersects the circle. So our region of integration is simply a circular wedge of radius a and angle β from the x-axis. Then our integral is much easier in polar coordinates:

$$\int_0^a \int_0^\beta \ln{(r^2)} r \ d\theta \ dr = \beta \int_0^a \ln{(r^2)} r \ dr$$

With a substitution $u = r^2$, this integral can be done with integration by parts, or looked up in a table, with the final answer being $\beta a^2 (\ln{(a)} - \frac{1}{2})$.