(1) Find the average value of the function $1 + 3x + y$ on the triangle with vertices 
$(0, 0), (1, 0),$ and $(0, 2)$.

Solution: The area of the triangle is equal to one, so the average value will simply be the integral

$$
\int_0^1 \int_0^{2-2x} (1 + 3x + y) \, dy \, dx = \int_0^1 (4 - 4x^2) \, dx = \frac{8}{3}
$$

(2) Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cylinder $x^2 + y^2 = 1$.

Solution: This is probably easiest in cylindrical coordinates. Solving the sphere boundary equation for $z$ we find $z = \pm \sqrt{9 - x^2 - y^2} = \pm \sqrt{9 - r^2}$. So the volume is

$$
\int_0^{2\pi} \int_1^3 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_1^3 2r \sqrt{9-r^2} \, dr \, d\theta
$$

$$
= \int_0^{2\pi} \frac{2}{3} (9 - r^2)^{3/2} \bigg|_1^3 d\theta = \frac{4\pi}{3} \frac{8^{3/2}}{3}
$$

(3) Compute the integral $\int \int \int_R \sqrt{x^2 + y^2} \, dV$ where $R$ is the region inside the cylinder $x^2 + y^2 = 25$ and between $z = -1$ and $z = 4$.

Solution: Again, cylindrical coordinates are the best choice for this problem.

$$
\int \int \int_R \sqrt{x^2 + y^2} \, dV = \int_0^5 \int_{-1}^{2\pi} \int_0^4 r^2 \, dz \, d\theta \, dr = \int_0^5 \int_0^{2\pi} 5r^2 \, d\theta \, dr
$$

$$
= \int_0^5 10\pi r^2 \, dr = \frac{10\pi r^3}{3} \bigg|_0^5 = \frac{1250\pi}{3}
$$

(4) Find the volume of the solid bounded by the planes $z = x, y = x, x + y = 2,$ and $z = 0$.

Solution: This is a tetrahedron. By considering any three of the four boundary equations, we can find that the vertices are $(0, 0, 0), (1, 1, 0), (1, 1, 1),$ and
(0, 2, 0). This helps to sketch the figure and determine the bounds for the integral:

\[ V = \int_0^1 \int_x^{2-x} \int_0^x dz \, dy \, dx = \int_0^1 \int_x^{2-x} x \, dy \, dx = \int_0^1 xy|_{x}^{2-x} \, dx \]

\[ = \int_0^1 (2x - 2x^2) \, dx = \frac{1}{3} \]

(5) Change the order of integration of \( \int_0^2 \int_{\arctan(x)}^{\arctan(\pi x)} \, dy \, dx \) and evaluate the integral.

Solution: This question is definitely a bit harder than one I would put on an exam. The integration region is shown below.

To do the \( x \)-integral first we need to split up the region into two pieces because of the corner at \((2, \arctan(2))\). Then we have

\[
\int_0^{\arctan(2)} \tan(y) \, dx \, dy + \int_{\arctan(2)}^{\arctan(2\pi)} \tan(y) / \pi \, dx \, dy = \\
\int_0^{\arctan(2)} (\tan(y) - \tan(y) / \pi) \, dy + \int_{\arctan(2)}^{\arctan(2\pi)} (2 - \tan(y) / \pi) \, dy = \\
-2 \arctan(2) + 2 \arctan(2\pi) + \frac{(-1 + \pi) \log(5)}{2\pi} + \frac{\log \left( \frac{5}{1+4\pi^2} \right)}{2\pi} \approx .827 \ldots
\]

The final answer has been simplified using several properties of the logarithm: \( \log(a) - \log(b) = \log(a/b) \), \( \log(1/b) = - \log(b) \), and \( \log(a^b) = b \log(a) \).

(6) Compute the integral \( \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2} \, dz \, dy \, dx \) by changing to cylindrical coordinates.
Solution: The projection of the region onto the x-y plane is the disk of radius 1. So the integral can be rewritten in cylindrical coordinates as:

\[ \int_{0}^{2\pi} \int_{0}^{1} \int_{r^2}^{2-r^2} r^4 \, dz \, dr \, d\theta \]

which evaluates to

\[ = \frac{1}{5} \int_{0}^{2\pi} \int_{0}^{1} (2 - 2r^2) r^4 \, dr \, d\theta = \frac{8\pi}{35} \]

(7) This difficult a problem would be extra credit: Assuming that \( \beta \in (0, \pi/2) \) and \( a > 0 \), compute the following integral

\[ \int_{0}^{a \sin \beta} \int_{\sqrt{a^2 - y^2}}^{\cot \beta} \int_{0}^{1} \ln (x^2 + y^2) \, dz \, dx \, dy \]

Solution: The z-integral is easy and we get

\[ \int_{0}^{a \sin \beta} \int_{\sqrt{a^2 - y^2}}^{\cot \beta} \ln (x^2 + y^2) \, dx \, dy \]

The first thing to do is understand the region of integration. The upper x-boundary \( x = \sqrt{a^2 - y^2} \) is the right-hand semicircle of radius \( a \) centered at \((0, 0)\). The lower x-boundary is the line \( x = \cot (\beta) y \) or \( y = \tan (\beta) x \), a line through \((0, 0)\) with angle \( \beta \). The y boundaries are the x-axis and the height where the line intersects the circle. So our region of integration is simply a circular wedge of radius \( a \) and angle \( \beta \) from the x-axis. Then our integral is much easier in polar coordinates:

\[ \int_{0}^{\alpha} \int_{0}^{\beta} \ln (r^2) r \, d\theta \, dr = \beta \int_{0}^{a} \ln (r^2) r \, dr \]

With a substitution \( u = r^2 \), this integral can be done with integration by parts, or looked up in a table, with the final answer being \( \beta a^2 (\ln (a) - \frac{1}{2}) \).