

Math 3298 Practice Final

This is roughly twice as long as the actual exam.

- (1) Reverse the order of integration for the integral $\int_0^1 \int_x^1 \int_0^{y^2} f(x, y, z) dz dy dx$.

Solution: $\int_0^1 \int_{\sqrt{z}}^1 \int_0^y f(x, y, z) dx dy dz$

- (2) Compute the vector line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the path $(4-3t, -2+2t, \pi t)$, $t \in [0, 1]$, and $\vec{F} = (2x \cos z - x^2, z - 2y, y - x^2 \sin z)$.

Solution: The field is conservative since $\text{curl}(\vec{F}) = \vec{0}$. A potential function for \vec{F} is $f = x^2 \cos z - x^3/3 + yz - y^2$, so $\int_C \vec{F} \cdot d\vec{r} = f(1, 0, \pi) - f(3, -2, 0) = -1/3 - (9 - 9 - 4) = 11/3$.

- (3) Find the linearization of $f(x, y)$ at $(x, y) = (0, 1)$ if $f = h(u(x, y), v(x, y))$ and $\text{grad}(h)|_{(1,1)} = (\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v})|_{(1,1)} = (2, 3)$, $u(x, y) = x + y$, and $v(x, y) = y^2$.

Solution: The linearization is $L(x, y) = f(0, 1) + \frac{\partial f}{\partial x}|_{(0,1)}(x-0) + \frac{\partial f}{\partial y}|_{(0,1)}(y-1)$. To find the partial derivatives we use the chain rule and the derivatives $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 1$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y}|_{0,1} = 2$:

$$\frac{\partial f}{\partial x}|_{(0,1)} = \frac{\partial h}{\partial u}|_{(1,1)} \frac{\partial u}{\partial x}|_{(0,1)} + \frac{\partial h}{\partial v}|_{(1,1)} \frac{\partial v}{\partial x}|_{(0,1)} = 2 \cdot 1 + 3 \cdot 0 = 2$$

$$\frac{\partial f}{\partial y}|_{(0,1)} = \frac{\partial h}{\partial u}|_{(1,1)} \frac{\partial u}{\partial y}|_{(0,1)} + \frac{\partial h}{\partial v}|_{(1,1)} \frac{\partial v}{\partial y}|_{(0,1)} = 2 \cdot 1 + 3 \cdot 2 = 8$$

So

$$L(x, y) = f(0, 1) + 2(x - 0) + 8(y - 1) = f(0, 1) + 2x + 8y - 8.$$

Since neither f nor h is given explicitly this is all that is possible to determine, besides the fact that $f(0, 1) = h(1, 1)$.

- (4) Find the surface area of the torus parameterized by $x = (2 + \cos(v)) \cos(u)$, $y = (2 + \cos(v)) \sin(u)$, $z = \sin(v)$, with $u \in [0, 2\pi]$ and $v \in [0, 2\pi]$.

Solution: The surface area element is computed from the length of the cross product of the partials $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$, where $\vec{r} = (x, y, z)$. After many uses of the

identity $\sin^2(t) + \cos^2(t) = 1$, this simplifies to

$$dS = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv = |2 + \cos(v)| du dv.$$

The absolute value function can be dropped because $2 + \cos(v) > 0$. Now the surface area can be computed:

$$S.A. = \int_0^{2\pi} \int_0^{2\pi} (2 + \cos(v)) du dv = 2\pi \int_0^{2\pi} (2 + \cos(v)) dv = 8\pi^2.$$

- (5) Find the maxima and minima of $f(x, y) = \frac{1}{x} + \frac{2}{y}$ on the set $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

Solution: I will use the Lagrange multiplier method. Let $g = \frac{1}{x^2} + \frac{1}{y^2} - 1$ (the constraint), and then require that $\nabla(f) = \lambda \nabla(g)$, i.e.

$$\left(-\frac{1}{x^2}, -\frac{2}{y^2}\right) = \left(-\frac{2\lambda}{x^3}, -\frac{2\lambda}{y^3}\right).$$

After clearing denominators we find that $\lambda = x/2 = y$. Using that relation between x and y , the constraint equations becomes $g(x, x/2) = \frac{1}{x^2} + \frac{4}{x^2} - 1 = 0$ or $x^2 = 5$. So there are two critical points on the constraint curve, $\pm(\sqrt{5}, \sqrt{5}/2)$. Comparison with other values of f on the constraint curve shows that $f(\sqrt{5}, \sqrt{5}/2) = \sqrt{5}$ is a maximum and $f(-\sqrt{5}, -\sqrt{5}/2) = -\sqrt{5}$ is a minimum.

- (6) Find the volume of the solid wedge bounded by the planes $z = 0$ and $z = -2y$ and the cylinder $x^2 + y^2 = 4$ (with $y \geq 0$).

Solution: In cylindrical coordinates

$$\begin{aligned} V &= \int_0^\pi \int_0^2 \int_{-2r \sin \theta}^0 dz r dr d\theta = \int_0^\pi \int_0^2 2r^2 \sin \theta dr d\theta \\ &= \int_0^\pi 16 \sin \theta / 3 d\theta = 16/3. \end{aligned}$$

- (7) Use Green's Theorem to find the smooth, simple, closed and positively oriented curve in the plane for which the line integral $\oint \left(\frac{x^2 y}{4} + \frac{y^3}{3}\right) dx + x dy$ has the largest possible value.

Solution: The corresponding double integral from Green's theorem is $\int \int_R (1 - x^2/4 - y^2) dA$. The integrand is positive in the interior of the ellipse $x^2/4 + y^2 = 1$, so we choose this ellipse as the desired curve ($x = 2 \cos(t)$, $y = \sin(t)$).

- (8) Compute the value of $\int \int_S (\nabla \times \vec{F}) \cdot \vec{n} dS$ where S is the upper half of the ellipsoid $4x^2 + 9y^2 + 36z^2 = 36$, $z \geq 0$, with upward pointing normal, and $\vec{F} = (y, x^2, (x^2 + y^2)^{3/2} e^{xyz})$.

Solution: The presence of nasty stuff like e^{xyz} inspires us to reformulate the computation using Stokes' theorem. The ellipse boundary can be parameterized as $x = 3 \cos t$, $y = 2 \sin t$. The corresponding line integral is then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (2 \sin t, 9 \cos^2 t) \cdot (-3 \sin t, 2 \cos t) dt = -6\pi.$$

- (9) Let $\vec{r}(t)$ be a curve in space with unit tangent, normal, and binormal vectors \vec{T} , \vec{N} , and \vec{B} . Show that $\frac{d\vec{B}}{dt}$ is perpendicular to \vec{T} .

Solution: Since $\vec{B} = \vec{T} \times \vec{N}$, $\frac{d\vec{B}}{dt} = \frac{d\vec{T}}{dt} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{dt}$. However \vec{N} is parallel to $\frac{d\vec{T}}{dt}$, so in fact $\frac{d\vec{B}}{dt} = \vec{T} \times \frac{d\vec{N}}{dt}$. Finally, note that \vec{a} is always perpendicular to $\vec{a} \times \vec{b}$ for any vectors \vec{a} , \vec{b} .

- (10) Compute the flux integral $\int \int_S \vec{F} \cdot \vec{n} dS$ where S is the graph of $z = 1 - x^2 - y^2$, with upward normal, for $z \geq 0$, and with $\vec{F} = (xz, yz, 2z^2)$.

Solution: The given flux integral can be computed directly or by using the divergence theorem. I will do it both ways for comparison (on a test, one method would be sufficient). Directly: since the surface is given as a graph $z = f(x, y)$, $\vec{n} dS = d\vec{S} = (-f_x, -f_y, 1) = (2x, 2y, 1)$. Then

$$\begin{aligned} \int \int_S \vec{F} \cdot \vec{n} dS &= \int \int_S (xz, yz, 2z^2)|_S \cdot (2x, 2y, 1) dx dy \\ &= \int \int_S (1-x^2-y^2)(2x^2+2y^2)+2(1-x^2-y^2) dx dy = 2 \int_0^{2\pi} \int_0^1 (1-r^2)(2r^2+2(1-r^2))r dr d\theta \\ &= 4 \int_0^{2\pi} \int_0^1 (r - r^3) dr d\theta = \pi. \end{aligned}$$

We cannot immediately apply the divergence theorem because this surface is not closed. However, we could consider the closed surface $S_2 = S \cup S_1$, where S_1

is the unit disk $z = 0$, $x^2 + y^2 \leq 1$ and with normal $(0, 0, -1)$. Since $\vec{F} = 0$ on S_1 , this addition doesn't actually affect the flux integral, i.e. $\int \int_{S_2} \vec{F} \cdot \vec{n} \, dS_2 = \int \int_S \vec{F} \cdot \vec{n} \, dS$ but $\int \int_{S_2} \vec{F} \cdot \vec{n} \, dS_2 = \iiint \operatorname{div} \vec{F} \, dV$. Since $\operatorname{div} \vec{F} = 6z$, the value we are after can be computed in cylindrical coordinates as:

$$\int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6zr \, dzdrd\theta = 3 \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} (1-r^2)^2 r \, drd\theta = \pi.$$