Homework 5, due Friday June 27

(1) Sketch the vector field \( \vec{F} = (y, 1) \) in the square \( x \in [-2, 2], \ y \in [-2, 2] \).

(2) Sketch the vector field \( \vec{F} = (x - y)\vec{i} + x\vec{j} \) in the square \( x \in [-2, 2], \ y \in [-2, 2] \).

(3) Match the vector field plots on the third page to the following vector fields (as usual the \( x \)-axis is the horizontal axis in these plots):
   (I) \( \vec{F} = (x - 1, x + 3) \)
   (II) \( \vec{F} = (y, x) \)
   (III) \( \vec{F} = (\sin(x), 1) \)
   (IV) \( \vec{F} = (y, 1/x) \)

(4) Compute the scalar line integral \( \int_C y \ ds \) where \( C \) is the curve \( x = y^2 \) for \( y \in [0, 3] \).

(5) Compute the scalar line integral \( \int_C x/y \ ds \) where \( C \) is the curve \( x = t^3, \ y = t^4, \ t \in [1/2, 1] \).

(6) Compute the scalar line integral \( \int_C xy^3 \ ds \) where \( C \) is the curve \( x = 3\sin(t), \ y = 3\cos(t), \ z = 4t \) for \( t \in [0, \pi/2] \).

(7) Is the vector line integral \( \int_C \vec{F} \cdot d\vec{s} \) positive, negative, or zero for the \( \vec{F} \) and \( C \) shown below? (The curve \( C \) is in blue.)

(8) Compute the vector line integral \( \int_C \vec{F} \cdot d\vec{s} \) where \( \vec{F} = (yz, xz, xy) \) and \( C \) is the curve \( \vec{r}(t) = (t, t^2, t^3) \) for \( t \) in \( [0, 1] \).
(9) Find a function $f$ such that $\nabla f = \vec{F}$ where $\vec{F} = (4xy + \ln(x), 2x^2)$. Use this to evaluate $\int_C \vec{F} \cdot d\vec{s}$ where $C$ is a curve beginning at $(1,0)$ and ending at $(2,2)$ along which $x > 0$.

(10) Use Green’s Theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (\sqrt{x} + y^3, x^2 + \sqrt{y})$ and $C$ is the graph of $y = \sin(x)$ from $(0,0)$ to $(\pi,0)$ and the line segment from $(\pi,0)$ to $(0,0)$.

(11) Use Green’s Theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (e^x + x^2y, e^y - xy^2)$ and $C$ is the circle $x^2 + y^2 = 16$ (oriented positively).

(12) Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ directly and with Green’s Theorem if $\vec{F} = (xy^2, x^3)$, and $C$ is the rectangle (oriented positively) $(0,0), (2,0), (2,3)$, and $(0,3)$.

(13) Show that the area of a polygon with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ (in counterclockwise order) is

$$A = \frac{1}{2} \left[ (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n) \right]$$

by using Green’s theorem and calculating the line integral of $(-y/2, x/2)$ along each line segment.

(14) Calculate the curl and divergence of the vector field $\vec{F} = (2, x + yz, xy - \sqrt{z})$.

(15) Calculate the curl and divergence of the vector field $\vec{F} = (\cos(yz), 0, -\sin(xy))$.

(16) Construct a potential function for the vector field $(e^z, 6, xe^z)$.

(17) Is there a vector field $\vec{G}$ such that $\nabla \times \vec{G} = (xy^2, yz^2, zx^2)$? Explain why or why not.

(18) Prove that $\nabla \cdot (f \vec{F}) = f \nabla \cdot \vec{F} + \vec{F} \cdot \nabla f$ if $f$ is a scalar function of $(x,y,z)$ with continuous partial derivatives and $\vec{F}$ is a vector field whose components also have continuous partial derivatives.

(19) If $\vec{r} = (x, y, z)$, $r = |\vec{r}|$, and $\vec{F} = \vec{r}/r^p$ ($p$ is a real number), compute $\nabla \cdot \vec{F}$. Is there a value of $p$ for which $\nabla \cdot \vec{F} = 0$?

(20) Parameterize the plane which contains the point $(1,1,1)$ and which is parallel to the vectors $(1,0,-1)$ and $(1,-1,0)$.

(21) Parameterize the lower half of the ellipsoid $x^2/9 + y^2/4 + z^2 = 1$.

(22) Compute the surface area of the part of the plane $3x + 2y + z = 6$ that is inside the cylinder $x^2 + y^2 = 16$. 