Math 4230 Practice final exam solutions.

(1) Find all the complex solutions to the equation $\cos(z) = \sin(z)$.

Solution: From the definition of the complex $\sin(z)$ and $\cos(z)$, we are looking for solutions to

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{2i} = \sin(z)$$

Since $e^{-iz} = 1/e^{iz}$, we can rewrite this equation in terms of $q = e^{iz}$ as

$$\frac{q + 1/q}{2} = \frac{q - 1/q}{2i}$$

Since $q \neq 0$, we can multiply by q and regroup terms to get $q^2(1-i) = 1+i$, or $q^2 = \frac{1+i}{1-i} = i$. So q must be one of the two square roots of i, which are $\pm\frac{1+i}{\sqrt{2}} = \pm\sqrt{2}e^{i\pi/4} = \{e^{i\pi/4}, e^{i5\pi/4}\}$.

So either $q = e^{iz} = e^{i\pi/4}$ or $q = e^{iz} = e^{i5\pi/4}$. Inverting the exponential is not unique - we have to include any multiple of $2\pi ni$ for $n \in \mathbb{Z}$. So $iz = i\pi/4 + i2\pi n$ or $iz = i5\pi/4 + i2\pi n$. So z is either $\pi/4$ or $5\pi/4$ plus any integer multiple of 2π. In other words, the only solutions are the ones you would expect from the real trig functions (somewhat surprising).

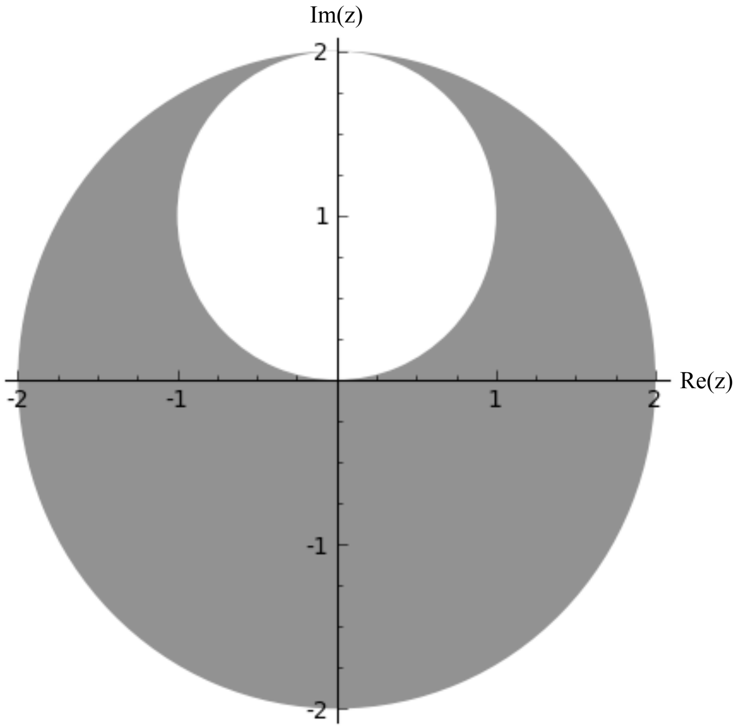
(2) Without computing the integral exactly, show that

$$\left| \int_\Gamma \frac{1}{z^2 - i} \, dz \right| < \frac{\pi}{3}$$

if Γ is the counter-clockwise arc of the circle $|z| = 2$ starting at 2 and ending at $2i$.

Solution: The integral can be bounded by the product of the length of the contour times the maximum modulus of the integrand on the contour. Since this contour is a quarter of a circle of radius 2, the length is $(2\pi 2)/4 = \pi$. The integrand is largest when the modulus of the denominator $z^2 - i$ is smallest, which occurs when the argument of z is $\pi/4$ and $z^2 = 4i$. At that point $|\frac{1}{z^2-i}| = 1/3$. So the bound is $\pi/3$. This is a strict inequality because $|\frac{1}{z^2-i}| < 1/3$ at every other point on the quarter circle.

(3) Find an analytic function that maps the shaded region below onto the upper half-plane. (One way is to compose a Mobius transformation with an exponential function.)



Solution: Using the hint strategy, we first map the cusp point $2i$ to ∞ using a Mobius transformation. While we are at it, we might as well choose to send 0 to 0 and -2 to $i\pi$ so that the image is the same strip we looked at in homework 10. Having made three choices there is only one such Mobius function $f_1(z) = \frac{az+b}{cz+d}$. Our conditions become

$$f(2i) = \infty : \text{ same as } 1/f(2i) = 0, \text{ or } \frac{c2i+d}{a2i+b} = 0,$$
$$f(0) = b/d = 0,$$

and

$$f(-2) = \frac{-2a+b}{-2c+d} = i\pi$$

So $b = 0$ and $d = -2ic$ from the first two conditions. If we choose $c = 1$, then $d = -2i$ and the third condition can be rearranged to give $-2a = i\pi(-2-2i)$ and then $a = -\pi + i\pi$. So $w = f_1(z) = \pi z(\frac{-1+i}{z-2i})$.

This transformation maps the shaded region into the strip $0 < Im(w) < \pi$. The exponential function sends this to the upper half plane, so the full transformation we want is

$$e^{f_1(z)} = e^{\pi z(\frac{-1+i}{z-2i})}.$$

There are other possible answers, since we could compose at the end with any analytic function that takes the upper half-plane to itself.

(4) Find the recursion relation for the coefficients of the power series solution $\sum_{n=0}^{\infty} a_n z^n$ around $z = 0$ which satisfies the differential equation

$$g'' - zg' - g = 0, \; g(0) = 0, \; g'(0) = 1.$$

Solution: We use a series form $g = \sum_{n=0}^{\infty} a_n z^n$, and then $-zg' = \sum_{n=1}^{\infty} -na_n z^n$ and $g'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$. So

$$g'' - zg' - g = 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - na_n + a_n\right] z^n = 0$$

Each coefficient of every power of z must vanish, so the sum of the constants $2a_2 + a_0 = 0$, and for $n \geq 1$ we must have $(n+2)(n+1)a_{n+2} - na_n + a_n = 0$ or

$$a_{n+2} = a_n \frac{n-1}{(n+2)(n+1)} \quad \text{(for } n \geq 1\text{)}.$$

Since $g(0) = 0$, $a_0 = 0$, which implies that all the even coefficients vanish: $a_{2n} = 0$.

(5) Construct a function $f(z)$ which has a pole of order 2 at the origin, a simple pole at ∞, and $Res(f;0) = 1$.

Solution: Any function of the form $\frac{A}{z^2} + \frac{1}{z} + B + Cz$ with constants $A \neq 0$, B, $C \neq 0$ has the required properties.

(6) Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+9)^2}\, dx$ by using the residue theorem.

Solution: Since $(z^2+9)^2 = (z-3i)^2(z+3i)^2$, the integrand has a pair of second-order poles at $z = 3i$ and $z = -3i$. For an integral of this type (covered in section 6.3 of our text) we can use the limit of semi-circular contours in the upper half plane, which means we only need the residue at $z = 3i$.

The residue can be calculated from the formula (in 6.1):

$$Res(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}} \left[(z - z0)^m f(z)\right].$$

In our case, $z_0 = 3i$ and $m = 2$, so

$$Res\left(\frac{1}{(z^2+9)^2}; 3i\right) = \lim_{z \to 3i} \frac{d}{dz}\left[(z-3i)^m \frac{1}{(z^2+9)^2}\right] = \frac{d}{dz}\left[\frac{1}{(z+3i)^2}\right]\Bigg|_{z=3i}$$

$$= \frac{-2}{(6i)^3} = -i/108$$

The integral is $2\pi i$ times this residue, so

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+9)^2}\, dx = (2\pi i)(-i/108) = \pi/54.$$