Math 4230 Practice Midterm Solutions

(1) What complex numbers z, if any, have the property that $z^i = 1$?

Solution: z^i is defined as $e^{ilog(z)}$ where the complex $log(z)$ is multivalued (the imaginary part is only determined up to a multiple of $i2\pi$; in other words $log(z) = Log(z) + i2\pi k$ for some integer k). Writing $1 = e^{i2\pi m}$ (where m is an integer) this implies that for $z^i = 1$, we need $i(Log(z) + i2\pi k) = i(Log(|z|) + iArg(z) + i2\pi k) = i2\pi m$. So the imaginary part of $Log(|z|) + iArg(z) + i2\pi k$ (for integer k) must be zero. This means $Arg(z) = -2\pi k$, which is equivalent to z being real and positive. We must also have $Log(|z|) = 2\pi m$ for integer m. Exponentiating, we get that $z = e^{2\pi m}$ for any integer m.

(2) If we define a single-valued complex inverse-tangent function as

$$Arctan(z) = \frac{i}{2} Log \frac{1 - iz}{1 + iz},$$

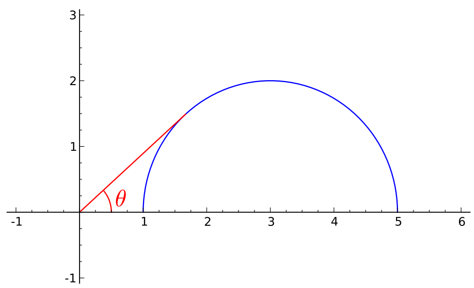
where is this function analytic? Explain your answer.

Solution: The rational function $\frac{1-iz}{1+iz}$ is analytic everywhere except when its denominator vanishes, at $z = i$. The Log function fails to be analytic on its branch cut, where its argument is real and non-positive. So we need to know for which z is $w = \frac{1-iz}{1+iz}$ real and nonpositive. Solving for z as a function of w gives us $z = i\frac{w-1}{w+1}$. For $w \in (-\infty, -1)$, this gives $z = iy$ where $y > 1$. For $w \in (-1, 0]$, $z = -iy$ for $y \geq 0$. So $Arctan(z)$ is analytic everywhere but the two rays $z = \pm iy$ for $y \geq 1$.

(3) Find a bound for the magnitude of the integral $\int_\Gamma (1 + Log(z))^3 \, dz$ if Γ is the upper semicircle of radius 2 centered at 3, starting at 5 and ending at 1.

Solution: The upper semicircle of radius 2 has length 2π. To get a bound for the integral, we also need to know the maximum magnitude of $(1 + Log(z))^3$. Since $|(1 + Log(z))^3| \leq (1 + |Log(z)|)^3$, we need a bound for $|Log(z)|$ on the contour. $|Log(z)| = \sqrt{Log(|z|)^2 + Arg(z)^2} \leq \sqrt{Log(5)^2 + A}$ where A is the maximum value of $Arg(z)$ on the semicircle. This is $Arcsin(2/3) \approx .73 < 3/4$ (see figure) so one bound would be

$$|\int_\Gamma (1 + Log(z))^3 \, dz| < 2\pi(1 + \sqrt{Log(5)^2 + (3/4)^2})^3 \approx 134.4$$



(4) Evaluate the contour integral
$$\int_\Gamma \frac{2z-1}{z^2-z-2}\ dz$$
if Γ is the curve $\gamma(t) = 12e^{i2\pi t} - e^{i6\pi t}$, $t \in [0,1]$.

Solution: It is much easier to deform the contour into a pair of connected unit circles (a "barbell" contour) around the poles (the zeros of the denominator of the integrand) and use the partial fraction decomposition $\frac{2z-1}{z^2-z-2} = \frac{1}{z+1} + \frac{1}{z-2}$. The poles are $z=-1$ and $z=2$. Then we have

$$\int_\Gamma \frac{2z-1}{z^2-z-2}\ dz = \int_{|z+1|=1} (\frac{1}{z+1} + \frac{1}{z-2})dz + \int_{|z-2|=1} (\frac{1}{z+1} + \frac{1}{z-2})dz$$
$$= i2\pi 1 + 0 + 0 + i2\pi 1 = i4\pi.$$

(The zero values from Cauchy's integral theorem).

(5) Is the function $1 - e^{x^2-y^2}\cos(2xy)$ the real part of an analytic function? Why or why not?

Solution: This is the real part of an analytic function. One way to see this is noting that it is the real part of $1 - e^{z^2}$. Another way is to check that it is harmonic - it satisfies the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.