Math 4326 Practice Final

The actual test will consist of 8 questions which should be fairly similar to some of the questions below. You will be required to answer 6 of those 8 questions.

(1) If the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$, compute the determinant

$$\begin{vmatrix} 2a+d-g & 2b+e-h & 2c+f-i \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix}.$$

(2) Determine a value for x such that the matrix $A = \begin{pmatrix} 1 & -1 & -2 & -2 \\ 0 & 2 & x & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is diagonalizable.

(3) Find the eigenvalues of the $n \times n$ matrices A with $a_{ij} = 1$ for $i \neq j$ and $a_{ii} = 2$.

(4) Indicate why each statement below is true or false:
   (a) If the kernel of a 7×6 matrix A is 3-dimensional, then the range of A must be 4-dimensional.
   (b) If A is 2×2, $tr(A) = 0$, and $det(A) = 1$ then A must be diagonalizable.
   (c) In the space of continuous real-valued functions on the interval $[-1, 1]$, the set of functions S such that $f(-x) = -f(x)$ is a vector subspace.
   (d) If A is a 4×4 matrix that can be partitioned into a 2×2 block-upper-triangular form $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ with A_1 and A_2 invertible, then A is invertible.

(5) Suppose that a matrix A can be factored as $A = BC$ where C is a square matrix. How are the column spaces of A, B, and C related?

(6) Suppose T is the linear transformation that sends polynomials of degree 3 or less into $\mathbb{R}^4$ by $T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ \frac{dp}{dx}(0) \\ \frac{dp}{dx}(1) \end{pmatrix}$. Find the matrix for T relative to the basis $\{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ and the standard basis for $\mathbb{R}^4$. What is the rank of this transformation?

(7) Find the general solution to the linear differential equation $x' = Ax$ where $A = \begin{pmatrix} 17 & -24 \\ 12 & -7 \end{pmatrix}$.

(8) Show that if A is a diagonalizable $n \times n$ matrix and $c_A(\lambda)$ is its characteristic polynomial that $c_A(A) = 0$.

(9) Find the condition number (the ratio of the largest to the smallest singular value) of the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$.

(10) Indicate why each statement below is true or false. If it is false, find additional conditions that would make it true.

(a) For any non-zero column vector $v \in \mathbb{R}^n$, vv^T is a projection matrix.

(b) If A is a symmtric matrix, then the matrix product $B^T AB$ is symmetric as well.

(c) An orthogonal projection matrix can have eigenvalues 0, 1, and -1.

(d) If W is a subspace of $\mathbb{R}^n$, let $z = proj_W(y)$ be the orthogonal projection of y onto W. Then $proj_W(z) = z$.

(11) Suppose that A is an $n \times n$ invertible symmetric positive definite matrix. Show that A^{-1} is also positive definite.

(12) Use the QR decomposition $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

to solve the least-squares problem of minimizing $|Ax - b|$ where $b = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$.