Math 4326 Practice Final Solutions

Please notify me if you think these solutions have mistakes.

The actual test will consist of 8 questions which should be fairly similar to some of the questions below. You will be required to answer 6 of those 8 questions.

(1) If the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$, compute the determinant

$$\begin{vmatrix} 2a+d-g & 2b+e-h & 2c+f-i \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix}.$$

Solution: Adding multiples of one row to another does not affect the determinant, so

$$\begin{vmatrix} 2a+d-g & 2b+e-h & 2c+f-i \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix} = \begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix}.$$

Multiplying a row by a constant changes the determinant by that constant. Since each of the three rows has been multiplied by 2, we have

$$\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix} = 2^3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2^3 * 5 = 40.$$

A common mistake is to think that $|cA| = c|A|$ for a scalar c and $n \times n$ matrix A, when in fact $|cA| = c^n|A|$.

(2) Determine a value for x such that the matrix $A = \begin{pmatrix} 1 & -1 & -2 & -2 \\ 0 & 2 & x & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is diagonalizable.

Solution: There are two eigenvalues (1 and 2) each with algebraic multiplicity two. To be diagonalizable, each eigenspace must be two-dimensional, which means that the nullspace of $A - \lambda$ must be two-dimensional. If we first examine the $\lambda = 2$ eigenspace, we see that

$$A - 2I = \begin{pmatrix} -1 & -1 & -2 & -2 \\ 0 & 0 & x & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and this will have two pivots (and thus have a $4-2=2$ dimensional nullspace) only if $x=2$. Likewise, considering

$$A - I = \begin{pmatrix} 0 & -1 & -2 & -2 \\ 0 & 1 & x & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

shows that $x=2$ is the only choice that will give a two-dimensional nullspace. So $x=2$ is the only solution.

(3) Find the eigenvalues of the $n \times n$ matrices A with $a_{ij} = 1$ for $i \neq j$ and $a_{ii} = 2$.

Solution: This could be done by induction or a direct argument. I will show the direct argument here.

The eigenvalues of A can be computed by finding the solutions of the characteristic equation $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & \dots & 1 & 1 \\ 1 & 2-\lambda & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 2-\lambda & 1 \\ 1 & 1 & \dots & 1 & 2-\lambda \end{vmatrix} = 0.$

The determinant is unaffected by adding a multiple of a row to other rows, or a multiple of a column to other columns. We can use this property to try to rearrange the determinant to an upper- or lower-triangular form. If we subtract the first row from all the rows below it, we have

$$\begin{vmatrix} 2-\lambda & 1 & \dots & 1 & 1 \\ 1 & 2-\lambda & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 2-\lambda & 1 \\ 1 & 1 & \dots & 1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & \dots & 1 & 1 \\ \lambda-1 & 1-\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda-1 & 0 & \dots & 1-\lambda & 0 \\ \lambda-1 & 0 & \dots & 0 & 1-\lambda \end{vmatrix}$$

Now if we add each column except the first to the first column we get

$$\begin{vmatrix} 2-\lambda & 1 & \dots & 1 & 1 \\ \lambda-1 & 1-\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda-1 & 0 & \dots & 1-\lambda & 0 \\ \lambda-1 & 0 & \dots & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1+n-\lambda & 1 & \dots & 1 & 1 \\ 0 & 1-\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-\lambda & 0 \\ 0 & 0 & \dots & 0 & 1-\lambda \end{vmatrix}$$

Since this is upper triangular we can see that the characteristic equation is $(1-\lambda)^{n-1}(1+n-\lambda) = 0$ and the eigenvalues are 1, with multiplicity $n-1$, and $1+n$ with multiplicity 1. (Since the matrix is symmetric, the algebraic multiplicities are equal to the geometric multiplicities.)

(4) Indicate why each statement below is true or false:

(a) If the kernel of a 7×6 matrix A is 3-dimensional, then the range of A must be 4-dimensional.

Solution: False. The range is 3-dimensional, since for a $m \times n$ matrix $dim(range) + dim(kernel) = n$.

(b) If A is 2×2, $tr(A) = 0$, and $det(A) = 1$ then A must be diagonalizable.
Solution:
True (over the complex numbers). We can determine the eigenvalues since $tr(A) = \lambda_1 + \lambda_2 = 0$ and $det(A) = \lambda_1 \lambda_2 = 1$. The first condition shows $\lambda_2 = -\lambda_1$, which we can substitute in to the second condition to find $\lambda_1^2 = -1$. So $\lambda_1 = i$ and $\lambda_2 = -i$. Since the eigenvalues are distinct, A is diagonalizable over complex matrices. It is not diagonalizable over only real matrices, although it must be similar to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over real matrices.

(c) In the space of continuous real-valued functions on the interval $[-1, 1]$, the set of functions S such that $f(-x) = -f(x)$ is a vector subspace.
Solution: True. Suppose that f_1 and f_2 are in S. We must show that a linear combination g of f_1 and f_2 is still in S:

$$g(-x) = (c_1 f_1 + c_2 f_2)(-x) = c_1 f_1(-x) + c_2 f_2(-x)$$

$$= -(c_1 f_1(x) + c_2 f_2(x)) = -g(x).$$

(d) If A is a 4×4 matrix that can be partitioned into a 2×2 block-upper-triangular form $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ with A_1 and A_2 invertible, then A is invertible.
Solution: False. It is true that if A_1 and A_3 are invertible then A is invertible.

(5) Suppose that a matrix A can be factored as $A = BC$ where C is a square matrix. How are the column spaces of A, B, and C related?
Solution: The column space of A must be contained in the column space of B, since if $x \in Col(A)$ then $x = Ay = BCy = B(Cy)$. If C is invertible, then this containment can be reversed by considering $AC^{-1} = B$ and then the two column spaces are equal.

(6) Suppose T is the linear transformation that sends polynomials of degree 3 or less into $\mathbb{R}^4$ by $T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ \frac{dp}{dx}(0) \\ \frac{dp}{dx}(1) \end{pmatrix}$. Find the matrix for T relative to the basis $\{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ and the standard basis for $\mathbb{R}^4$. What is the rank of this transformation?

Solution: The columns of the matrix will be the images of the basis of the domain. So we compute

$$T(1) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ T(x) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ T(x^2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \ T(x^3) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

Then the matrix is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. This row-reduces to the identity matrix which shows the rank is 4.

(7) Find the general solution to the linear differential equation $x' = Ax$ where $A = \begin{pmatrix} 17 & -24 \\ 12 & -7 \end{pmatrix}$.

Solution: The characteristic polynomial of A is $|\lambda I - A| = \lambda^2 - 10 * \lambda + 169$. The roots of this are the eigenvalues $\lambda = 5 \pm 12i$. We can choose either eigenvalue, so arbitrarily lets choose $5 + 12i$. Its eigenvector is found by computing the kernel of $A - (5+12i)I$. To do this we row-reduce $A - (5+12i)I = \begin{pmatrix} 12 - 12i & -24 \\ 12 & -12i - 12 \end{pmatrix}$ to find that $v_1 = \begin{pmatrix} 2 \\ 1 - i \end{pmatrix}$. This means that the general complex solution to the differential equation is $z_1 v_1 e^{5t+12it} + z_2 \bar{v}_1 e^{5t-12it}$ where z_1 and z_2 are complex constants. The general real solution is

$$x(t) = c_1 Re(v_1 e^{5t+12it}) + c_2 Im(v_1 e^{5t+12it}) =$$

$$c_1 \begin{pmatrix} 2e^{5t} \cos 12t \\ e^{5t} \sin 12t + e^{5t} \cos 12t \end{pmatrix} + c_2 \begin{pmatrix} 2e^{5t} \sin(12t) \\ e^{5t} \sin(12t) - e^{5t} \cos(12t) \end{pmatrix}.$$

Different choices of the eigenvalue and eigenvector can lead to somewhat different looking answers.

(8) Show that if A is a diagonalizable $n \times n$ matrix and $c_A(\lambda)$ is its characteristic polynomial that $c_A(A) = 0$.

Solution: The key property to note is that powers of diagonalizable matrices are easy to compute: $A^n = (SDS^{-1})^n = SD^nS^{-1}$. If $c_A(x) = a_0 + a_1 x + \ldots + a_n x^n$ then

$$c_A(A) = a_0 I + a_1 A + \ldots + a_n A^n$$

$$= a_0 SIS^{-1} + a_1 SDS^{-1} + \ldots + a_n SD^nS^{-1} = S(a_0 I + a_1 D + \ldots + a_n D^n)S^{-1}.$$

Since D is diagonal with the eigenvalues as diagonal entries,

$$c_A(A) = S \begin{pmatrix} c_A(\lambda_1) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & c_A(\lambda_n) \end{pmatrix} S^{-1} = S0S^{-1} = 0.$$

(9) Find the condition number (the ratio of the largest to the smallest singular value) of the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution: We compute the singular values as the positive square roots of the eigenvalues of $A^TA = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$. The characteristic polynomial of A^TA is $det(\lambda I - A^TA) = \lambda^2 - 4\lambda + 2$, which has roots $2 \pm \sqrt{2}$. So $\sigma_1 = \sqrt{2+\sqrt{2}}$ and $\sigma_2 = \sqrt{2-\sqrt{2}}$, and the condition number is $\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = 1 + \sqrt{2}$. The last simplification is somewhat tricky and would not be required.

(10) Indicate why each statement below is true or false. If it is false, find additional conditions that would make it true.

(a) For any non-zero column vector $v \in \mathbb{R}^n$, vv^T is a projection matrix.
Solution: False. vv^T is a projection matrix only if the length of v is 1. But $\frac{v\ v^T}{v^Tv}$ is always a projection matrix.

(b) If A is a symmtric matrix, then the matrix product B^TAB is symmetric as well.
Solution: True. $(B^TAB)^T = B^TA^T(B^T)^T = B^TAB$.

(c) An orthogonal projection matrix can have eigenvalues 0, 1, and -1.
Solution: False. An orthogonal projection matrix P has the properties that $P^2 = P$ (it is a projection) and $P^T = P$ (it is symmetric). Since it is symmetric, it can be diagonalized $P = SDS^{-1}$, and since $P = P^2$ we have $SDS^{-1} = SD^2S^{-1}$ and then $D = D^2$. Since D is diagonal, we have $\lambda = \lambda^2$ for each entry on the diagonal which means $\lambda = 0$ or $\lambda = 1$. So the eigenvalues can only be 0 or 1.

(d) If W is a subspace of $\mathbb{R}^n$, let $z = proj_W(y)$ be the orthogonal projection of y onto W. Then $proj_W(z) = z$.
Solution: True. z is already in W, so it is its own projection onto W.

(11) Suppose that A is an $n \times n$ invertible symmetric positive definite matrix. Show that A^{-1} is also positive definite.

Solution: Since A is symmetric, it can be orthogonalized diagonalized, and since it is positive definite the eigenvalues must be positive. So $A = UDU^T$ where U is an orthogonal matrix and D is diagonal with positive entries on the diagonal. Then we can compute $A^{-1} = UD^{-1}U^T$ and see that it is also symmetric and positive definite.

(12) Use the QR decomposition $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ to solve the least-squares problem of minimizing $|Ax - b|$ where $b = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: We can use a QR decomposition $A = QR$ to solve the least-squares problem by solving $Rx = Q^T b$. Since $Q^T b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ this is particularly easy in this case - the unique least-squares solution is $x = 0$. (This is because in this case b is orthogonal to the column space of A.)