Math 4326 Practice Midterm 1 Solutions

This practice test should be roughly twice as long as the real one; some of the problems may be harder than on the exam. You need to justify all your answers.

The test will cover chapters 1 through 4.7, but not 2.6.

(1) Describe the set of b for which the equation $Ax = b$ has a solution if

$$A = \begin{pmatrix} 3 & 17 & 0 \\ -6 & -34 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: The row-echelon form augmented coefficient matrix is:

$$\begin{pmatrix} 3 & 17 & 0 & b_1 \\ -6 & -34 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{pmatrix} \sim \begin{pmatrix} 3 & 17 & 0 & b_1 \\ 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & b_2 + 2b_1 \end{pmatrix}$$

which is only consistent if $b_2 + 2b_1 = 0$. So there is a two-dimensional subspace $B \subset R^3$ for which $Ax = b$ will have a solution if $b \in B$. A basis for B is $\{(0, 0, 1)^T, (1, -2, 0)^T\}$.

(2) Find bases for the nullspace and column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Solution: The easiest way to determine the nullspace (kernel) and column space is to compute the reduced row-echelon form of A. We did this in class, it is:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Since this has two pivots, the rank of A (the dimension of the column space) is two, and one way to choose a basis is take columns from A from the pivot columns - i.e. in this case $(1, 4, 7)^T$ and $(2, 5, 8)^T$.

A basis for the nullspace can be found from the reduced row-echelon form by extending it to the augmented form for a homogeneous system $Ax = 0$, i.e.

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and then we can see that for any $x \in N(A)$, $x_2 + 2x_3 =$ and $x_1 - x_3 = 0$. x_3 is a free variable. By writing $x = x_3(1, -2, 1)^T$ we see that $(1, -2, 1)^T$ is a basis for $N(A)$.

(3) Let A be the matrix $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. Construct a 4×2 matrix D using only 1s and 0s as entries such that $AD = I$. Is it possible that $CA = I$ for some 4×2 matrix C? Why or why not?

Since $d_{21} + d_{31} + d_{41} = 0$ and $d_{12} + d_{22} + d_{32} = 0$ and we are only supposed to use 0 and 1, those entries must be 0 and $d_{11} = 1$, $d_{42} = 1$.

It is not possible to find such a C. There are many ways to see this. One way is considering the ranks of A and C. Since A has rank 2, CA cannot have a rank greater than 2, but the 4×4 identity matrix has rank 4.

(4) Prove that the system

$$\begin{aligned} 3x_1 + 3x_2 + 4x_4 &= b_1 \\ x_1 + x_3 - x_4 &= b_2 \\ -x_2 + x_3 + x_4 &= b_3 \end{aligned}$$

always has a solution. You should not explicitly find the solution set.

Solution: There are several ways to do this. One way is to reduce the augmented coefficient matrix to row-echelon form, and note that there are three pivots. Another way is to compute the determinant of the coefficient matrix A, and note that since it is nonzero (its 16) A is invertible and the unique solution is $A^{-1}(b_1, b_2, b_3)^T$.

(5) The numbers 20604, 53227, 25755, 20927, and 78421 are divisible by 17. Explain why the determinant $det \begin{pmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{pmatrix}$ is also divisible by 17.

Solution: If we add 10000 times column 1 to column 5, 1000 times column 2 to column 5, 100 times column 3 to column 5, and 10 times column 4 to column 5, the determinant of the resulting matrix is the same as that of the original matrix. Every entry of the fifth column of this new matrix is divisible by 17. Every term in the determinant involves a entry from each column, in particular column 5, so the determinant is divisible by 17.

You should not do such a problem by actually computing the determinant.

(6) Suppose that v_1 and v_2 are linearly independent column vectors in $\mathbb{R}^4$. What are the possible values of the rank of the matrix $v_1v_1^T + v_2v_2^T$?

Solution: This problem will be easier after we learn about projections and inner-products. The rank must be 2, but to prove that you have to use the Cauchy-Schwarz inequality (or something equivalent to it) to see that $(v_1^Tv_2)^2 < (v_1^Tv_1)(v_2^Tv_2)$. It would suffice at this stage to note that the rank must be between 1 and 2. It is at least 1 since applying the matrix to v_1 or v_2 gives a nonzero linear combination of them. It is at most 2 since everything in the column space is a combination of v_1 and v_2.

(7) Extra credit: Let $\mathbb{P}_2$ be the vector space of polynomials in the variable x with coefficients in $\mathbb{R}$ and degree at most 2. Find the matrix representation of the linear operator $T : \mathbb{P}_2 \to \mathbb{P}_2$ where $T(p) = \frac{dp}{dx} + p$ for $p \in \mathbb{P}_2$, with respect to the basis $\{1, x, x^2\}$.

Solution: In that basis, the matrix representing T is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$