Math 4326 Practice Midterm 2

This practice test should be approximately twice as long as the real one; some of the problems may be harder than on the exam. You need to justify all your answers.

The test will cover chapters 1 through 5, with an emphasis on chapter 5.

(1) Determine a value for h such that the matrix $A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 5 & h & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is diagonalizable.

(2) Find the eigenvalues of an $n \times n$ matrices A with $a_{ij} = 1$ for every i and j (all entries are 1). What is the rank of this matrix?

(3) Indicate why each statement below is true or false:
   (a) A is a 7×7 matrix with two distinct eigenvalues, and one of the eigenspaces is 5-dimensional. Then A must be diagonalizable.
   (b) A nilpotent matrix A ($A^m = 0$ for some positive integer m) can only have one eigenvalue.
   (c) By Gerschgorin's theorem, the matrix $\begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 10 & 5 \\ 1 & 1 & 1 & 20 \end{pmatrix}$ must be diagonalizable.

(4) Find an invertible matrix P and a matrix C of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ such that $\begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix} = PCP^{-1}$

(5) Suppose T is the linear transformation that sends polynomials of degree 3 or less into $\mathbb{R}^2$ by $T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$. Find the matrix for T relative to the basis $\{1, x, x^2, x^3\}$ for $\mathbb{P}$ and the standard basis for $\mathbb{R}^2$.

(6) Solve the initial value problem $x' = Ax$ where $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ and $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(7) Extra credit: Show that if A is a diagonalizable $n \times n$ matrix and $c_A(\lambda)$ is its characteristic polynomial that $c_A(A) = 0$.