Math 4326 Practice Midterm 2 Solutions

Please alert me if you think there is a typo in these solutions.

(1) Determine a value for h such that the matrix $A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 5 & h & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is diagonalizable.

Solution:

The eigenvalues are 2, 5, and 1. For A to be diagonalizable, the kernel of $A - 2I$ needs to be two-dimensional. $A - 2I$ row-reduces to

$$\begin{pmatrix} 0 & 3 & 4 & 5 \\ 0 & 3 & h & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 3 & 4 & 0 \\ 0 & 0 & h-4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so we need $h = 4$.

(2) Find the eigenvalues of an $n \times n$ matrices A with $a_{ij} = 1$ for every i and j (all entries are 1). What is the rank of this matrix?

Solution:

The only eigenvalues are n (with algebraic and geometric multiplicity 1) and 0 (with algebraic and geometric multiplicity $n-1$). There are many ways to see that; one way to find the algebraic multiplicities is to calculate the characteristic polynomial by induction. The rank of the matrix is 1 - one way to see this is to write A as an outer product $A = vv^T$ where $v^T = (1, \dots, 1)$. Any outer product of vectors has rank 1.

(3) Indicate why each statement below is true or false:

(a) A is a 7×7 matrix with two distinct eigenvalues, and one of the eigenspaces is 5-dimensional. Then A must be diagonalizable.

Solution:

False. The second eigenspace must be two-dimensional and it does not have to be.

(b) A nilpotent matrix A ($A^m = 0$ for some positive integer m) can only have one eigenvalue.

Solution:

True. The only possible eigenvalue is 0. If there were a nonzero eigenvalue λ with eigenvector v, then $A^m v = \lambda^m v \neq 0$ but this must be 0 if A^m is the zero matrix.

(c) By Gerschgorin's theorem, the matrix $\begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 10 & 5 \\ 1 & 1 & 1 & 20 \end{pmatrix}$ must be diagonalizable.

Solution:

True. The circles of Gerschgorin are disjoint, which means that there is one eigenvalue per circle, so the matrix has 4 distinct eigenvalues and is diagonalizable.

(4) Find an invertible matrix P and a matrix C of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ such that $\begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix} = PCP^{-1}$.

Solution:

The characteristic equation of the matrix is $(-2-\lambda)(2-\lambda)+5 = \lambda^2+1 = 0$, so the eigenvalues are $\pm i$. Now we make an arbitrary choice between the eigenvalues and choose $\lambda = i$. Row-reducing $A - iI$ to find its kernel (the i-eigenspace) we get the complex eigenvector $v^T = (1, \frac{2+i}{5})$. The columns of the matrix P we desire are the real and imaginary parts of the eigenvector -

$$P = \begin{pmatrix} 1 & 0 \\ 2/5 & 1/5 \end{pmatrix}$$

which gives

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(5) Suppose T is the linear transformation that sends polynomials of degree 3 or less into $\mathbb{R}^2$ by $T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$. Find the matrix for T relative to the basis $\{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ and the standard basis for $\mathbb{R}^2$.

Solution:

The columns of the matrix are the images of $\{1, x, x^2, x^3\}$, so the matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

(6) Solve the initial value problem $x' = Ax$ where $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ and $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Solution:

The eigenvalues of A are 1 and -1, with eigenvectors $v_1 = (3, -1)^T$ and $v_2 = (1, -1)^T$ respectively. The general solution is $x = c_1 v_1 e^t + c_2 v_2 e^{-t}$. The initial condition is $(0, 1)^T = c_1 v_1 + c_2 v_2$, which is a linear system with the unique solution $c_1 = 1/2$, $c_2 = -3/2$. So

$$x(t) = \begin{pmatrix} \frac{3}{2}e^t - \frac{3}{2}e^{-t} \\ \\ \frac{-1}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}$$

(7) Extra credit: Show that if A is a diagonalizable $n \times n$ matrix and $c_A(\lambda)$ is its characteristic polynomial that $c_A(A) = 0$.

Solution: This is a special case of the Cayley-Hamilton theorem, which states the same result for any $n \times n$ matrix. For the diagonalizable case one wants to write $A = SDS^{-1}$ and use the fact that $A^m = SD^mS^{-1}$.