Homework Assignment 3 for Math 5327. Due Friday, February 15th.

(i) Read 3.1 - 3.6, 4.1 - 4.4, and 4.7.

(ii) Ungraded problems; you do not need to hand these in, but I will assume that you have done them and that you will ask questions if there is something you do not completely understand:

3.2.6, 3.5.6, 3.5.7, 3.6.7, 4.1.2.

(1) Consider the vector space $\mathcal{Q}_2[x, y, z]$ of polynomials with real coefficients in variables x, y, and z which have total degree ≤ 2, i.e.

$$\mathcal{Q}_2[x, y, z] = \left\{ \sum a_{ijk} x^i y^j z^k \mid i + j + k \leq 2,\ i, j, k \geq 0,\ a_{ijk} \in \mathbb{R} \right\}$$

(for example, $x^2 + y^2 - 3yz - 2y$ is in $\mathcal{Q}_2[x, y, z]$.)

For $p \in \mathcal{Q}_2[x, y, z]$, let $d_k(p) = \dfrac{\partial p}{\partial k}(0, 0, 0)$ where $k \in \{x,\ y,\ z\}$. Similarly, let d_{kl} be defined as $\dfrac{\partial^2 p}{\partial k \partial l}(0, 0, 0)$. Since Clairaut's theorem says that $\frac{\partial^2 p}{\partial k \partial l} = \frac{\partial^2 p}{\partial l \partial k}$, $d_{kl} = d_{lk}$.

(a) Are the set of operators d_k and d_{kl} linearly independent?

(b) Do the set of operators d_k and d_{kl} span the dual space $\mathcal{Q}_2^*[x, y, z]$ (the space of linear functions on $\mathcal{Q}_2[x, y, z]$, also denoted as $\mathcal{L}(\mathcal{Q}_2[x, y, z], \mathbb{R})$) ?

(2) Let P_n be the space of polynomials in x with real coefficients and degree $\leq n$. If $[a, b]$ is a real interval then for $p \in P_n$

$$\int_a^b p(x)\ dx = \sum_{i=1}^{n+1} m_i p(t_i)$$

where the m_i are constants independent of p and the t_i are distinct points. This is called a quadrature formula.

(a) Prove that such a quadrature formula always exists.

(b) For $n = 3$, $a = -1$, $b = 1$, and $t_1 = -c$, $t_2 = -d$, $t_3 = d$, $t_4 = c$ find a formula for the constants m_i. When are the m_i all positive?