Homework Assignment 8 for Math 5327. Due Wednesday, May 1st, by 3pm.

Read sections 7.9 and 7.11.

Ungraded problems: 7.7.7, 7.7.9, 7.8.5, 7.9.1, 7.11.4.

(1) Find an 11×11 matrix L such that $\dim(N(L)) = 4$, $\dim(N(L^2)) = 7$, $\dim(N(L^3)) = 9$, $\dim(N(L^4)) = 10$, $\dim(N(L^5)) = 11$.

(2) Find the Jordan normal form of the matrix $\begin{pmatrix} -2 & 0 & 1 \\ -1 & -2 & 2 \\ -1 & -1 & 1 \end{pmatrix}$.

(3) Compute the dimensions of the quotient spaces $N_i = N(L^i)/N(L^{i-1})$ for $1 \leq i \leq 5$ if

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(4) If a matrix $A \in \mathbb{C}^{n \times n}$ is Hermitian ($A^* = A$), show that $B = (I - iA)(I + iA)^{-1}$ is unitary ($B^*B = I$). (You can assume that $(I + iA)$ is invertible.)

(5) Optional, extra credit: Consider the differential equation $\mathcal{L}y = 0$ where $\mathcal{L}$ is the linear first-order differential operator $\mathcal{L} = \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_1\frac{d}{dx} + a_0$. Denote the set of solutions within the space of complex functions $y(x)$ on $\mathbb{R}$ with n continuous derivatives by V. There is an existence theorem in the theory of differential equations which guarantees that V is a n-dimensional space.

We can think of $\mathcal{L}$ as $p(D)$ where $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0 = \prod_{i=1}^{n}(z - r_i)$. Assuming that all of the r_i are distinct, compute the Jordan form of $\mathcal{D}|_V$ where $\mathcal{D}$ is the differentiation operator $\frac{d}{dx}$.