Practice final for Math 5327. This is longer than the actual exam.

(1) Find the Jordan normal form of the matrix $\begin{pmatrix} 1 & -1 & -1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$. You do not have to explicitly compute the similarity matrix.

(2) Show that a $n \times n$ matrix A is normal if and only if $\mathrm{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2$ where λ_i is the i^{th} eigenvalue of A.

(3) Show that the eigenvalues of the tridiagonal Toeplitz matrices

$$B_n = \begin{pmatrix} a & b & 0 & 0 & \dots & 0 & 0 \\ c & a & b & 0 & \dots & 0 & 0 \\ 0 & c & a & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a & b \\ 0 & 0 & 0 & 0 & \dots & c & a \end{pmatrix}$$

only depend on a and the product bc (instead of b and c individually).

(4) Is the set of trace-free (i.e. their trace is zero) linear transformations of $\mathbb{C}^n$ a vector space? Justify your answer.

(5) Find the orthogonal projection matrix that projects onto the subspace of $\mathbb{R}^4$ satisfying the conditions $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 - x_2 = 0$.

(6) What is the set of λ for which $\lim_{n\to\infty} J_\lambda^n = 0$ if $J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$? Extra credit question: for which λ in this set does the 2-norm of J_λ^n decrease monotonically?

(7) Find the least squares solution to $\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(8) Suppose that a 8×8 matrix A has the property that $Range(A^4) \subset Nullspace(A^3)$, and $\dim(Range(A^3)) = 3$. What are the possible values of the rank of A? What are the possible Jordan forms for A?

(9) Suppose P is a projection matrix on a finite-dimensional vector space V, and the range of P is the subspace $W \subset V$. If $\{w_1, w_2, \ldots, w_m\}$ is a basis for W, and $\{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_n\}$ is a basis for V, prove that the set

$$\{w_1, w_2, \ldots, w_m, v_1 - P(v_1), v_2 - P(v_2), \ldots, v_n - P(v_n)\}$$

is also a basis for V.

(10) Find the singular value decomposition of the matrix $A = \begin{pmatrix} 2 & -1 \\ 3 & 6 \end{pmatrix}$. Your answer should be kept exact (not numerical).

(11) Extra credit (the exam will have at least one extra credit problem): if P_M and P_N are orthogonal projectors onto subspaces M and N, respectively, show that the orthogonal projection onto $M \cap N$ can be written as $2P_M(P_M + P_N)^\dagger P_N$ where † denotes the pseudo-inverse.