Practice midterm for Math 5327. A sheet of notes and calculator will be allowed on the exam. This is much longer than the actual exam, which will be 4 to 5 questions.

(1) Describe the set of solutions to the system

$$\begin{pmatrix} -1 & 0 & 1 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(2) Determine the matrix $[S]_{\beta\beta'}$ where S is the linear transformation $S : P_2 \to P_3$ (P_j is the vector space of polynomials of degree $\leq j$) such that $S(p) = \int_0^t p(x)dx - tp(t)$ and $\beta = \{1, t, t^2\}$, $\beta' = \{1, t, t^2, t^3\}$.

(3) For x and y column vectors in $\mathbb{R}^n$, consider the matrix $A = I + xy^T$ (sometimes called a rank-one perturbation of the identity). Show that if A is invertible then $A^{-1} = I + \alpha xy^T$ for some real number α. Also, find a formula for α.

(4) Use induction to compute the determinant of matrices of the form

$$B_n = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

for $n \geq 2$. B_n is an n by n matrix.

(5) Suppose that a 6×6 matrix A has the property that $Range(A^3) = Nullspace(A^3)$. What are the possible values of the rank of A?

(6) Find the QR decomposition of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$.

(7) Suppose A is a 3×3 matrix whose SVD $U\Sigma V^*$ has the following properties: $U = V$, and $\sigma_1 = 2$, $\sigma_2 = 1$, $\sigma_3 = \frac{1}{2}$. What is the best rank-2 approximation to A? Is it an orthogonal projection?

(8) Show that we can define an inner product on $\mathbb{R}^{n \times n}$ by $(A, B) = tr(A^*B)$.

(9) Find the orthogonal projection matrix that projects onto the subspace $x_1 + 2x_2 + 3x_3 = 0$ of $\mathbb{R}^3$ (with the standard inner product).

(10) Construct the first Householder reflection matrix H_1 for $A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{pmatrix}$ - i.e. H_1 should be a unitary matrix and the first column of $H_1 A$ should be $(\sqrt{14}, 0, 0)^T$.

(11) Find the least squares solution to $\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.