Practice midterm solutions for Math 5327. A sheet of notes and calculator will be allowed on the exam. This is much longer than the actual exam, which will be 4 to 5 questions.

If you find a typo or mistake in these solutions please let me know.

(1) Describe the set of solutions to the system

$$\begin{pmatrix} -1 & 0 & 1 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Solution: The coefficient matrix A has rank 2, and so does the augmented matrix since the middle column of A is a multiple of the vector on the right-hand side. This means there is a 1-dimensional subspace of solutions, which in parametric form is $(0, -1, 0) + t(1, 0, 1)$.

(2) Determine the matrix $[S]_{\beta\beta'}$ where S is the linear transformation $S : P_2 \to P_3$ (P_j is the vector space of polynomials of degree $\leq j$) such that $S(p) = \int_0^t p(x)dx - tp(t)$ and $\beta = \{1, t, t^2\}$, $\beta' = \{1, t, t^2, t^3\}$.

Solution: First we compute the value of S on each basis element of β. These values are $p(1) = 0$, $p(t) = -t^2/2$, and $p(t^2) = -2t^3/3$. The columns of $[S]_{\beta\beta'}$ are the coefficients of these values when expressed in terms of the β' basis, namely

$$[S]_{\beta\beta'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

(3) For x and y column vectors in $\mathbb{R}^n$, consider the matrix $A = I + xy^T$ (sometimes called a rank-one perturbation of the identity). Show that if A is invertible then $A^{-1} = I + \alpha xy^T$ for some real number α. Also, find a formula for α.

Solution: Just expand $(I + xy^T)(I + \alpha xy^T)$ and note that $xy^T xy^T$ is just xy^T multiplied by the scalar $y^T x$. Then we can solve for $\alpha = -\frac{1}{1+y^T x}$. This will always be defined if A is invertible since if $y^T x = -1$ then $Ax = 0$ which is only possible for invertible A if x is zero. But if $x = 0$ then $y^T x = 0$.

(4) Use induction to compute the determinant of matrices of the form

$$B_n = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

for $n \geq 2$. B_n is an n by n matrix.

Solution: Expanding along the top row, we can compute that $Det(B_n) = -Det(B_{n-1}) - Det(B_{n-2})$. Since $Det(B_2) = 0$ and $Det(B_3) = 1$, we see that $Det(B_4) = -1$. This results in a period-3 sequence $0, 1, -1, 0, 1, -1, \dots$. In (awkward) closed form $Det(B_n) = \lfloor 3(\frac{n-1}{3} - \lfloor \frac{n-1}{3} \rfloor) - 1 \rfloor$.

(5) Suppose that a 6×6 matrix A has the property that $Range(A^3) = Nullspace(A^3)$. What are the possible values of the rank of A?

(6) Find the QR decomposition of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$.

Solution: This could be done with either of the Gram-Schmidt or Householder methods. I will use the modified Gram-Schmidt method.

The first column is already a unit vector, so we can set $q_1 = a_1$ and $r_{11} = 1$. Subtracting off the q_1 component from $v_2 = a_2$ and $v_3 = a_3$ gives $r_{12} =< q_1|a_2 >= 1$ and $r_{13} =< q_1|a_3 >= 1$, so $v_2^{(2)} = v_2 - r_{12}q_1 = (0, 2, 2)^T$ and $v_3^{(2)} = v_3 - r_{13}q_1 = (0, 2, 3)^T$.

Since $r_{22} = |v_2^{(2)}| = 2\sqrt{2}$, $q_2 = v_2^{(2)}/r_{22} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$ and subtract off the q_2 component from $v_3^{(2)}$: $r_{23} =< q_2|v_3^{(2)} >= \frac{5}{\sqrt{2}}$ so $v_3^{(3)} = v_3^{(2)} - r_{23}q_2 = (0, -\frac{1}{2}, \frac{1}{2})^T$.

Finally $r_{33} = |v_3^{(3)}| = \frac{1}{\sqrt{2}}$ and $q_3 = v_3^{(3)}/r_{33} = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$. So

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2\sqrt{2} & \frac{5}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is the QR decomposition.

(7) Suppose A is a 3×3 matrix whose SVD $U\Sigma V^*$ has the following properties: $U = V$, and $\sigma_1 = 2$, $\sigma_2 = 1$, $\sigma_3 = \frac{1}{2}$. What is the best rank-2 approximation to A? Is it an orthogonal projection?

Solution: The best rank-2 approximation will be $P = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^*$. It is not even a projection, so it is not an orthogonal projection. In order to be a projection, we would need at least $\sigma_1 = \sigma_2 = 1$ and $span(u_1, u_2) = span(v_1, v_2)$.

(8) Show that we can define an inner product on $\mathbb{R}^{n\times n}$ by $(A, B) = tr(A^* B)$.

Solution: The function is symmetric since $tr(AB) = tr(BA)$, and linear since $tr(\lambda A + B) = \lambda tr(A) + tr(B)$ and by the linearity of matrix multiplication. Finally, note that $tr(A^*A) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \geq 0$ and equality holds only for the zero matrix. Note that we could have used $tr(A^T B)$ instead since the matrices are real - however, as written this is also a Hermitian inner product for complex matrices.

(9) Find the orthogonal projection matrix that projects onto the subspace $x_1 + 2x_2 + 3x_3 = 0$ of $\mathbb{R}^3$ (with the standard inner product).

Solution: The projection P should have a kernel $= span((1, 2, 3))$, so $P = I - \frac{vv^*}{v^*v}$ where $v = (1, 2, 3)^T$. This works out to

$$P = \begin{pmatrix} \frac{13}{14} & -\frac{1}{7} & -\frac{3}{14} \\ -\frac{1}{7} & \frac{5}{7} & -\frac{3}{7} \\ -\frac{3}{14} & -\frac{3}{7} & \frac{5}{14} \end{pmatrix}$$

(10) Construct the first Householder reflection matrix H_1 for $A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{pmatrix}$ - i.e. H_1 should be a unitary matrix and the first column of $H_1 A$ should be $(\sqrt{14}, 0, 0)^T$.

Solution: If $a_1 = (1, 2, 3)^T$ then the difference vector $v = a_1 - |a_1| e_1 = (1 - \sqrt{14}, 2, 3)^T$. The first Householder reflector is

$$H_1 = I - 2\frac{vv^*}{v^*v} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \sqrt{\frac{2}{7}} & \frac{3}{\sqrt{14}} \\ \sqrt{\frac{2}{7}} & \frac{1}{91}\left(63 - 2\sqrt{14}\right) & -\frac{3}{91}\left(14 + \sqrt{14}\right) \\ \frac{3}{\sqrt{14}} & -\frac{3}{91}\left(14 + \sqrt{14}\right) & 1 + \frac{9}{-14+\sqrt{14}} \end{pmatrix}$$

and

$$H_1 A = \begin{pmatrix} \sqrt{14} & 9\sqrt{\frac{2}{7}} \\ 0 & \frac{2}{91}\left(-21 + 5\sqrt{14}\right) \\ 0 & \frac{1}{91}\left(28 + 15\sqrt{14}\right) \end{pmatrix}$$

(11) Find the least squares solution to $\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: This is a straightforward computation. You can compute the pseudoinverse of the coefficient matrix in a variety of ways (QR decomposition from Gram-Schmidt or Householder reflections, or the SVD). In exact arithmetic it suffices to just directly compute $A^\dagger = (A^*A)^{-1}A^*$ and then $x = A^\dagger b = (0, 1/3)$ (numerically this is usually a bad idea though).