Practice final solutions for Math 5327.

Please let me know if you find any typos or mistakes.

(1) Find the Jordan normal form of the matrix $\begin{pmatrix} -2 & 0 & 1 \\ -1 & -2 & 2 \\ -1 & -1 & 1 \end{pmatrix}$. You do not have to explicitly compute the similarity matrix.

Solution: The characteristic polynomial is $det(\lambda I - A) = x^3 + 3x^2 + 3x + 1 = (x+1)^3$. So the only eigenvalue is -1 with algebraic multiplicity 3. To check the geometric multiplicity, we examine the rank of

$$-I - A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix}$$

Since this has rank 2, the nullspace is 1 dimensional and there is 1 linearly independent eigenvector - i.e. the eigenvalue has geometric multiplicity 1. This already completely determines the Jordan form - there must be one 3×3 block, so

$$S^{-1}AS = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

(2) Show that a $n \times n$ matrix A is normal if and only if $trace(A^*A) = \sum_{i=1}^n |\lambda_i|^2$ where λ_i is the i^{th} eigenvalue of A (counted by algebraic multiplicity).

Solution: If A is normal, then we can unitarily diagonalize A as $A = UDU^*$ and $tr(A^*A) = tr(UD^*DU^*) = tr(D^*D) = \sum_{i=1}^n |\lambda_i|^2$.

For the other direction, every $n \times n$ matrix has a Schur decomposition $A = UTU^*$ where U is unitary and T is upper-triangular. Then $tr(A^*A) = tr(T^*T) = \sum_{i,j=1}^n |t_{ij}|^2 = \sum_{i=1}^n |\lambda_i|^2$. Since $t_{ii} = \lambda_i$, it follows that $\sum_i^n \sum_{j=i+1}^n |t_{ij}|^2 = 0$ and so T is diagonal and thus A is normal.

(3) Show that the eigenvalues of the tridiagonal Toeplitz matrices

$$B_n = \begin{pmatrix} a & b & 0 & 0 & \dots & 0 & 0 \\ c & a & b & 0 & \dots & 0 & 0 \\ 0 & c & a & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a & b \\ 0 & 0 & 0 & 0 & \dots & c & a \end{pmatrix}$$

depend on the product bc (instead of b and c individually).

Solution: This can be seen by deriving a recursive formula for the characteristic polynomial $p_n(\lambda) = det(\lambda I - B_n)$. By expanding the determinant along the top row, we see that $p_n(\lambda) = (\lambda - a)p_{n-1}(\lambda) - bc\ p_{n-2}(\lambda)$. Since $p_1 = (\lambda - a)$ and $p_2 = (\lambda - a)^2 - bc$, we see that only the product bc enters into the base cases and the recursion.

(4) Is the set of trace-free (i.e. their trace is zero) linear transformations of $\mathbb{C}^n$ a vector space? Justify your answer.

Solution: Yes, it is a vector space. Since the linear transformations of $\mathbb{C}^n$ are a vector space (of dimension n^2), we only need to check the closure of the set under addition and scalar multiplication. We can do this on any matrix representation of the linear transformations. If A and B are matrices with trace zero and s and t are scalars, then $tr(sA + tB) = s\ tr(A) + t\ tr(B) = 0$ by the linearity of the trace.

(5) Find the orthogonal projection matrix that projects onto the subspace of $\mathbb{R}^4$ satisfying the conditions $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 - x_2 = 0$.

Solution: By row reduction, or inspection, we can see that the solution set of the conditions can be written as span$\{w_1 = (1, 1, -2, 0), w_2 = (1, 1, 0, -2)\}$. If we orthonormalize these we get $v_1 = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, 0)$, $v_2 = (\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{-\sqrt{3}}{2})$. Now we can write the desired projection as

$$P = v_1 v_1^T + v_2 v_2^T = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

(6) What is the set of λ for which $\lim_{n\to\infty} J_\lambda^n = 0$ if $J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$? Extra credit question: for which λ in this set does the 2-norm of J_λ^n decrease monotonically?

Solution: Since $J_\lambda^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$, $\lim_{n\to\infty} J_\lambda^n = 0$ if and only if $|\lambda| < 1$.

For the extra credit part, we compute the 2-norm of J_λ^n by finding the square root of the largest eigenvalue w_n of $(J_\lambda^n)^T J_\lambda^n$. By considering the entries of J_λ^n, it can be shown that if $w_2 < w_1$ then the w_n will be monotonically decreasing. Equality between w_2 and w_1 occurs at the unique root of $2 - 4|\lambda| + 2|\lambda|^2 - 2|\lambda|^3 + |\lambda|^5 = 0$ in $(0, 1)$; numerically this is $|\lambda| \approx 0.589$.

(7) Find the least squares solution to $\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: Since the coefficient matrix A has maximal rank (2), we can compute the pseudoinverse as

$$A^\dagger = (A^TA)^{-1}A^T = \begin{pmatrix} \frac{1}{38} & \frac{4}{19} & \frac{7}{38} \\ -\frac{6}{19} & \frac{9}{19} & -\frac{4}{19} \end{pmatrix}$$

and $x = A^\dagger(1,1,1)^T = (\frac{8}{19}, \frac{-1}{19})$ is the least-squares solution.

(8) Suppose that a 8×8 matrix A has the property that $Range(A^4) \subset Nullspace(A^3)$, and $\dim(Range(A^3)) = 3$. What are the possible values of the rank of A? What are the possible Jordan forms for A?

Solution: First we find the possible Jordan forms, since the rank is easy to determine from that. The inclusion $Range(A^4) \subset Nullspace(A^3)$ tells us that A^7 must be the zero matrix. So the only eigenvalue is 0. In order for A^3 to be non-zero, there must be at least one Jordan block that is at least 4×4. If there were exactly two such blocks, $\dim(Range(A^3)) = 2$ so we can rule that out. So there must be one Jordan block that is at least 5×5, which means the smaller blocks will not contribute at all to $\dim(Range(A^3))$. So this larger block must be 6×6, and the only possible Jordan forms are:

$$S^{-1}AS = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \text{ or } \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

and from these we see the rank of A must be either 5 or 6.