Practice midterm 2 solutions for Math 5327. Please let me know if you think you have found a typo.

(1) Find the QR decomposition of \( A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix} \).

Solution: This could be done with either of the Gram-Schmidt or Householder methods. I will use the modified Gram-Schmidt method.

The first column is already a unit vector, so we can set \( q_1 = a_1 \) and \( r_{11} = 1 \). Subtracting off the \( q_1 \) component from \( v_2 = a_2 \) and \( v_3 = a_3 \) gives \( r_{12} = 1 \) and \( r_{13} = 1 \), so \( v_2 = v_2 - r_{12} q_1 = (0, 2, 2)^T \) and \( v_3 = v_3 - r_{13} q_1 = (0, 2, 3)^T \).

Since \( r_{22} = |v_2^{(2)}| = 2\sqrt{2}, q_2 = v_2^{(2)} / r_{22} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T \) and subtract off the \( q_2 \) component from \( v_3^{(2)} \): \( r_{23} = \langle q_2, v_3^{(2)} \rangle = \frac{5}{\sqrt{2}} \) so \( v_3 = v_3 - r_{23} q_2 = (0, -\frac{1}{2}, \frac{1}{2})^T \).

Finally \( r_{33} = |v_3^{(3)}| = \frac{1}{\sqrt{2}} \) and \( q_3 = v_3^{(3)} / r_{33} = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T \). So

\[
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2\sqrt{2} & \frac{5}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}
\]

is the QR decomposition.

(2) Suppose \( A \) is a \( 3 \times 3 \) matrix whose SVD \( U \Sigma V^* \) has the following properties:

\( U = V \), and \( \sigma_1 = 2, \sigma_2 = 1, \sigma_3 = \frac{1}{2} \). What is the best rank-2 approximation to \( A \)? Is it an orthogonal projection?

Solution: The best rank-2 approximation will be \( P = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* \). It is not even a projection, so it is not an orthogonal projection. In order to be a projection, we would need at least \( \sigma_1 = \sigma_2 = 1 \) and \( \text{span}(u_1, u_2) = \text{span}(v_1, v_2) \).

(3) Show that we can define an inner product on \( \mathbb{R}^{n \times n} \) by \( (A, B) = tr(A^*B) \).

Solution: The map is symmetric since \( tr(AB) = tr(BA) \) (example 3.6.5), and linear since \( tr(\lambda A + B) = \lambda tr(A) + tr(B) \) (example 3.3.1) and by the linearity of matrix multiplication. Finally, note that \( tr(A^*A) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \geq 0 \) and equality holds only for the zero matrix. Note that we could have used \( tr(A^T B) \) instead since the matrices are real - however, as written this is also a Hermitian inner product for complex matrices.

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(4) Find the orthogonal projection matrix that projects onto the subspace \( x_1 + 2x_2 + 3x_3 = 0 \) of \( \mathbb{R}^3 \) (with the standard inner product).

Solution: The projection \( P \) should have a kernel \( = \text{span}((1, 2, 3)) \), so \( P = I - \frac{vv^*}{v^*v} \) where \( v = (1, 2, 3)^T \). This works out to

\[
P = \begin{pmatrix}
\frac{13}{14} & -\frac{1}{7} & -\frac{3}{14} \\
-\frac{1}{7} & \frac{5}{7} & -\frac{3}{7} \\
-\frac{3}{14} & -\frac{3}{7} & \frac{5}{14}
\end{pmatrix}
\]

(5) Construct the first Householder reflection matrix \( H_1 \) for \( A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{pmatrix} \) - i.e. \( H_1 \) should be a unitary matrix and the first column of \( H_1 A \) should be \( (\sqrt{14}, 0, 0)^T \).

Solution: If \( a_1 = (1, 2, 3)^T \) then the difference vector \( v = a_1 - |a_1|e_1 = (1 - \sqrt{14}, 2, 3)^T \). The first Householder reflector is

\[
H_1 = I - 2\frac{vv^*}{v^*v} = \begin{pmatrix}
\frac{1}{\sqrt{14}} & \sqrt{\frac{2}{7}} & \frac{3}{\sqrt{14}} \\
\sqrt{\frac{2}{7}} & \frac{1}{91} (63 - 2\sqrt{14}) & -\frac{3}{91} (14 + \sqrt{14}) \\
\frac{3}{\sqrt{14}} & -\frac{3}{91} (14 + \sqrt{14}) & 1 + \frac{9}{14 + \sqrt{14}}
\end{pmatrix}
\]

and

\[
H_1 A = \begin{pmatrix}
\sqrt{14} & 9\sqrt{\frac{2}{7}} \\
0 & \frac{2}{91} (-21 + 5\sqrt{14}) \\
0 & \frac{1}{91} (28 + 15\sqrt{14})
\end{pmatrix}
\]

(6) Find the least squares solution to \( \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \).

Solution: This is a straightforward computation. You can compute the pseudoinverse of the coefficient matrix in a variety of ways (QR decomposition from Gram-Schmidt or Householder reflections, or the SVD). In exact arithmetic it suffices to just directly compute \( A^\dagger = (A^*A)^{-1}A^* \) and then \( x = A^\dagger b = (0, 1/3) \) (numerically this is usually a bad idea though).