Practice midterm 2 solutions for Math 5327. Please let me know if you think you have found a typo.

(1) Find the QR decomposition of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$.

Solution: This could be done with either of the Gram-Schmidt or Householder methods. I will use the modified Gram-Schmidt method.

The first column is already a unit vector, so we can set $q_1 = a_1$ and $r_{11} = 1$. Subtracting off the q_1 component from $v_2 = a_2$ and $v_3 = a_3$ gives $r_{12} =< q_1|a_2 >= 1$ and $r_{13} =< q_1|a_3 >= 1$, so $v_2^{(2)} = v_2 - r_{12}q_1 = (0,2,2)^T$ and $v_3^{(2)} = v_3 - r_{13}q_1 = (0,2,3)^T$.

Since $r_{22} = |v_2^{(2)}| = 2\sqrt{2}$, $q_2 = v_2^{(2)}/r_{22} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$ and subtract off the q_2 component from $v_3^{(2)}$: $r_{23} =< q_2|v_3^{(2)} >= \frac{5}{\sqrt{2}}$ so $v_3^{(3)} = v_3^{(2)} - r_{23}q_2 = (0, -\frac{1}{2}, \frac{1}{2})^T$.

Finally $r_{33} = |v_3^{(3)}| = \frac{1}{\sqrt{2}}$ and $q_3 = v_3^{(3)}/r_{33} = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$. So

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2\sqrt{2} & \frac{5}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is the QR decomposition.

(2) Suppose A is a 3×3 matrix whose SVD $U\Sigma V^*$ has the following properties: $U = V$, and $\sigma_1 = 2$, $\sigma_2 = 1$, $\sigma_3 = \frac{1}{2}$. What is the best rank-2 approximation to A? Is it an orthogonal projection?

Solution: The best rank-2 approximation will be $P = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^*$. It is not even a projection, so it is not an orthogonal projection. In order to be a projection, we would need at least $\sigma_1 = \sigma_2 = 1$ and $span(u_1, u_2) = span(v_1, v_2)$.

(3) Show that we can define an inner product on $\mathbb{R}^{n \times n}$ by $(A, B) = tr(A^*B)$.

Solution: The map is symmetric since $tr(AB) = tr(BA)$ (example 3.6.5), and linear since $tr(\lambda A + B) = \lambda tr(A) + tr(B)$ (example 3.3.1) and by the linearity of matrix multiplication. Finally, note that $tr(A^*A) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \geq 0$ and equality holds only for the zero matrix. Note that we could have used $tr(A^TB)$ instead since the matrices are real - however, as written this is also a Hermitian inner product for complex matrices.

(4) Find the orthogonal projection matrix that projects onto the subspace $x_1 + 2x_2 + 3x_3 = 0$ of $\mathbb{R}^3$ (with the standard inner product).

Solution: The projection P should have a kernel $= span((1,2,3))$, so $P = I - \frac{vv^*}{v^*v}$ where $v = (1,2,3)^T$. This works out to

$$P = \begin{pmatrix} \frac{13}{14} & -\frac{1}{7} & -\frac{3}{14} \\ -\frac{1}{7} & \frac{5}{7} & -\frac{3}{7} \\ -\frac{3}{14} & -\frac{3}{7} & \frac{5}{14} \end{pmatrix}$$

(5) Construct the first Householder reflection matrix H_1 for $A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{pmatrix}$ - i.e. H_1 should be a unitary matrix and the first column of $H_1 A$ should be $(\sqrt{14}, 0, 0)^T$.

Solution: If $a_1 = (1,2,3)^T$ then the difference vector $v = a_1 - |a_1| e_1 = (1 - \sqrt{14}, 2, 3)^T$. The first Householder reflector is

$$H_1 = I - 2\frac{vv^*}{v^*v} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \sqrt{\frac{2}{7}} & \frac{3}{\sqrt{14}} \\ \sqrt{\frac{2}{7}} & \frac{1}{91}\left(63 - 2\sqrt{14}\right) & -\frac{3}{91}\left(14 + \sqrt{14}\right) \\ \frac{3}{\sqrt{14}} & -\frac{3}{91}\left(14 + \sqrt{14}\right) & 1 + \frac{9}{-14+\sqrt{14}} \end{pmatrix}$$

and

$$H_1 A = \begin{pmatrix} \sqrt{14} & 9\sqrt{\frac{2}{7}} \\ 0 & \frac{2}{91}\left(-21 + 5\sqrt{14}\right) \\ 0 & \frac{1}{91}\left(28 + 15\sqrt{14}\right) \end{pmatrix}$$

(6) Find the least squares solution to $\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: This is a straightforward computation. You can compute the pseudoinverse of the coefficient matrix in a variety of ways (QR decomposition from Gram-Schmidt or Householder reflections, or the SVD). In exact arithmetic it suffices to just directly compute $A^\dagger = (A^* A)^{-1} A^*$ and then $x = A^\dagger b = (0, 1/3)$ (numerically this is usually a bad idea though).