5.12. (a) Yes. The poles $z = \pm j(0.9)$ are inside the unit circle so the system is stable.

(b) First, factor $H(z)$ into two parts. The first should be minimum phase and therefore have all its poles and zeros inside the unit circle. The second part should contain the remaining poles and zeros.

$$H(z) = \frac{1 + 0.2z^{-1}}{1 + 0.81z^{-2}} \quad \frac{1 - 9z^{-2}}{1}$$

minimum phase \quad \text{poles & zeros outside unit circle}

Allpass systems have poles and zeros that occur in conjugate reciprocal pairs. If we include the factor $(1 - \frac{1}{2}z^{-2})$ in both parts of the equation above the first part will remain minimum phase and the second will become allpass.

$$H(z) = \frac{(1 + 0.2z^{-1})(1 - \frac{1}{2}z^{-2})}{1 + 0.81z^{-2}} \frac{1 - 9z^{-2}}{1 - \frac{1}{2}z^{-2}}$$

$$= H_1(z)H_{ap}(z)$$

5.18. A minimum phase system with an equivalent magnitude spectrum can be found by analyzing the system function, and reflecting all poles are zeros that are outside the unit circle to their conjugate reciprocal locations. This will move them inside the unit circle. Then, all poles and zeros for $H_m(z)$ will be inside the unit circle. Note that a scale factor may be introduced when the pole or zero is reflected inside the unit circle.

(a) Simply reflect the zero at $z = 2$ to its conjugate reciprocal location at $z = \frac{1}{2}$. Then, determine the scale factor.

$$H_{min}(z) = 2 \left(1 - \frac{1}{2}z^{-1}\right)$$

(b) First, simply reflect the zero at $z = -3$ to its conjugate reciprocal location at $z = -\frac{1}{3}$. Then, determine the scale factor. This results in

$$H_{min}(z) = 3 \left(1 - \frac{1}{3}z^{-1}\right) \frac{(1 + \frac{1}{3}z^{-1})}{z^{-1} (1 + \frac{1}{3}z^{-1})}$$

The $(1 + \frac{1}{3}z^{-1})$ terms cancel, leaving

$$H_{min}(z) = 3 \left(1 - \frac{1}{3}z^{-1}\right) \frac{z^{-1}}{z^{-1}}$$

Note that the term $\frac{1}{1}$ does not affect the frequency response magnitude of the system. Consequently, it can be removed. Thus, the remaining term has a zero inside the unit circle, and is therefore minimum phase. As a result, we are left with the system

$$H_{min}(z) = 3 \left(1 - \frac{1}{2}z^{-1}\right)$$

(c) Simply reflect the zero at 3 to its conjugate reciprocal location at $\frac{1}{3}$ and reflect the pole at $\frac{1}{3}$ to its conjugate reciprocal location at $\frac{1}{3}$. Then, determine the scale factor:

$$H_{min}(z) = \frac{9 (1 - \frac{1}{3}z^{-1})(1 - \frac{1}{3}z^{-1})}{(1 - \frac{1}{3}z^{-1})^2}$$

5.24. (a) Taking the $z$-transform of both sides and rearranging

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{1}{1 + z^{-2}}$$

Since the poles and zeros (2 poles at $z = \pm 1/2$, 2 zeros at $z = \pm 2$) occur in conjugate reciprocal pairs the system is allpass. This property is easy to recognize since, as in the system above, the coefficients of the numerator and denominator $z$-polynomials get reversed (and in general conjugated).

(b) It is a property of allpass systems that the output energy is equal to the input energy. Here is the proof:

$$\sum_{n=-\infty}^{N-1} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |y[n]|^2$$

(by Parseval's Theorem)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})X(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

($|H(e^{j\omega})|^2 = 1$ since $h[n]$ is allpass)

$$= \sum_{n=-\infty}^{\infty} |x[n]|^2$$

(by Parseval’s theorem)

$$= \sum_{n=0}^{N-1} |x[n]|^2$$

$$= 5$$
5.71. (a) There are many possible solutions to this problem. The idea behind any solution is to have \( h[n] \) be an upsampled (by a factor of 2) version of \( g[n] \). That is, \( h[n] = \{ g[n/2], \; n = 0, \pm 2, \pm 4, ... \} \).

Thus, \( h[n] \) will process only the even-indexed samples. One such system would be described by

\[
\begin{align*}
\hat{h}[n] &= 2 \sin(0.5\pi(n - 10)) \\
h[n] &= \{ g[n/2], \; n = 0, \pm 2, \pm 4, ... \\
&= 1, \; \text{otherwise}
\end{align*}
\]

Thus, \( h[n] \) will process only the even-indexed samples. One such system would be described by

\[
\hat{h}[n] = \{ g[n/2], \; n = 0, \pm 2, \pm 4, ... \\
h[n] = \{ 1, \; \text{otherwise}
\]

(b) As in part a, there are many possible solutions to this problem. The idea behind any solution is to choose an \( h[n] \) that cannot be an upsampled (by a factor of 2) version of \( g[n] \). Clearly, choosing \( h[n] \) to filter odd-indexed samples satisfies this criterion. One such \( h[n] \) would be

\[
\hat{h}[n] = 1 + \delta[n - 2]
\]

(c) In general, the odd-indexed samples of \( h[n] \) must be zero, in order for a \( g[n] \) to be found for which \( r[n] = y[n] \). Thus, there must not be any odd powers of \( z^{-1} \) in \( H(z) \).

(d) For the conditions determined in part c, \( g[n] \) is a downsampled (by a factor of 2) version of \( h[n] \). That is, \( g[n] = h[2n] \).

5.75. Looking at the \( z \)-transform of the FIR filter,

\[
H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}
\]

Substituting \( m = N - 1 - n \) into the summation gives

\[
H(z) = \sum_{m=-N+1}^{N-1} h[m]z^{-m-N+1}
\]

Thus, for such a filter,

\[
H(1/z) = z^{N-1}H(z)
\]

If \( z_0 \) is a zero of \( H(z) \), then \( H(z_0) = 0 \), and

\[
H(1/z_0) = z_0^{N-1}H(z_0) = 0
\]

Consequently, even-symmetric linear phase FIR filters have zeros that are reciprocal images.