7.3. We are given the digital filter constraints

\[ 1 - \delta_1 \leq |H(e^{j\omega})| \leq 1 + \delta_1, \quad 0 \leq |\omega| \leq \omega_p \]
\[ |H(e^{j\omega})| \leq \delta_2, \quad \omega_s \leq |\omega| \leq \pi \]

and the analog filter constraints

\[ 1 - \delta_1 \leq |H_c(j\Omega)| \leq 1, \quad 0 \leq |\Omega| \leq \Omega_p \]
\[ |H_c(j\Omega)| \leq \delta_2, \quad \Omega_s \leq |\Omega| \]

(a) If we divide the digital frequency specifications by \((1 + \delta_1)\) we get

\[ 1 - \delta_1 = \frac{1 - \delta_1}{1 + \delta_1} \]
\[ \delta_1 = \frac{2\delta_1}{1 + \delta_1} \]
\[ \delta_2 = \frac{\delta_2}{1 + \delta_1} \]

(b) Solving the equations in Part (a) for \(\delta_1\) and \(\delta_2\), we find

\[ \delta_1 = \frac{\delta_1}{2 - \delta_1} \]
\[ \delta_2 = \frac{2\delta_2}{2 - \delta_1} \]

In the example, we were given

\[ \delta_1 = 1 - 0.89125 = 0.10875 \]
\[ \delta_2 = 0.17783 \]

Plugging in these values into the equations for \(\delta_1\) and \(\delta_2\), we find

\[ \delta_1 = 0.0575 \]
\[ \delta_2 = 0.1881 \]

The filter \(H'(z)\) satisfies the discrete-time filter specifications where \(H'(z) = (1 + \delta_1)H(z)\) and \(H(z)\) is the filter designed in the example. Thus,

\[
H'(z) = 1.0575 \left[ \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.5949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} \right]
\]
\[
= 0.3035 - 0.4723z^{-1} + \frac{0.3035 - 0.4723z^{-1} + 0.2570z^{-2}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.2660 + 1.2114z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}}
\]
\[
= 1.9624 - 0.6665z^{-1} + \frac{1.9624 - 0.6665z^{-1} + 0.2570z^{-2}}{1 - 1.2971z^{-1} + 0.6949z^{-2}}
\]

(c) Following the same procedure used in part (b) we find

\[
H'(z) = 1.0575 \left[ \frac{0.0007378(1 + z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})} \right]
\]
\[
\times \frac{1}{1 - 0.9044z^{-1} + 0.2155z^{-2}}
\]
\[
= \frac{0.0007802(1 + z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})}
\]
\[
\times \frac{1}{1 - 0.9044z^{-1} + 0.2155z^{-2}}
\]

7.7. Using the relation \(\omega = \Omega T\), the passband cutoff frequency, \(\omega_p\), and the stopband cutoff frequency, \(\omega_s\), are found to be

\[
\omega_p = 2\pi(1000)10^{-4}
\]
\[
= 0.2\pi \text{ rad}
\]
\[
\omega_s = 2\pi(1100)10^{-4}
\]
\[
= 0.22\pi \text{ rad}
\]

Therefore, the specifications for the discrete-time frequency response \(H_d(e^{j\omega})\) are

\[ 0.99 \leq |H_d(e^{j\omega})| \leq 1.01, \quad 0 \leq |\omega| \leq 0.20\pi \]
\[ |H_d(e^{j\omega})| \leq 0.01, \quad 0.22\pi \leq |\omega| \leq \pi \]
7.10. Using the bilinear transform frequency mapping equation,
\[ \omega_c = 2 \tan^{-1} \left( \frac{\Omega_c T}{2} \right) \]
\[ = 2 \tan^{-1} \left( \frac{2\pi(2000)(0.4 \times 10^{-3})}{2} \right) \]
\[ = 0.7589 \text{ rad} \]

7.11. Using the relation \( \omega = \Omega T \),
\[ \Omega_c = \frac{\omega_c}{T} \]
\[ = \frac{\pi}{4} \]
\[ = 0.0001 \]
\[ = 2500 \text{ rad} \]
\[ = 2\pi(1250) \text{ rad} \]

7.12. Using the bilinear transform frequency mapping equation,
\[ \Omega_c = \frac{2}{\pi} \tan \left( \frac{\omega_c}{2} \right) \]
\[ = \frac{2}{0.001} \tan \left( \frac{\pi/2}{2} \right) \]
\[ = 2000 \text{ rad} \]
\[ = 2\pi(318.3) \text{ rad} \]

7.15. This filter requires a maximal passband error of \( \delta_p = 0.05 \), and a maximal stopband error of \( \delta_s = 0.1 \). Converting these values to dB gives
\[ \delta_p = -26 \text{ dB} \]
\[ \delta_s = -20 \text{ dB} \]

This requires a window with a peak approximation error less than -26 dB. Looking in Table 7.1, the Hanning, Hamming, and Blackman windows meet this criterion.

Next, the minimum length \( L \) required for each of these filters can be found using the "approximate width of mainlobe" column in the table since the mainlobe width is about equal to the transition width. Note that the actual length of the filter is \( L = M + 1 \).

**Hanning:**
\[ 0.1\pi = \frac{8\pi}{M} \]
\[ M = 80 \]

**Hamming:**
\[ 0.1\pi = \frac{8\pi}{M} \]
\[ M = 80 \]

**Blackman:**
\[ 0.1\pi = \frac{12\pi}{M} \]
\[ M = 120 \]

7.16. Since filters designed by the window method inherently have \( \delta_1 = \delta_2 \) we must use the smaller value for \( \delta \).
\[ \delta = 0.02 \]
\[ A = -20 \log_{10}(0.02) = 33.9794 \]
\[ \beta = 0.5842(33.9794 - 21)^{0.4} + 0.07886(33.9794 - 21) = 2.65 \]
\[ M = \frac{A - 8}{2.285\Delta \omega} = \frac{33.9794 - 8}{2.285(0.65\pi - 0.63\pi)} = 180.95 \rightarrow 181 \]
7.34. (a) It is well known that convolving two rectangular windows results in a triangular window. Specifically, to get the $(M+1)$ point Bartlett window for $M$ even, we can convolve the following rectangular windows.

\[
\begin{align*}
r_1[n] &= \begin{cases} \sqrt{2/M}, & n = 0, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases} \\
r_2[n] &= r_1[n-1]
\end{align*}
\]

Using the known transform of a rectangular window we have

\[
W_{R_1}(e^{j\omega}) = \sqrt{\frac{2}{M}} \frac{\sin(\omega M/4)}{\sin(\omega/2)} e^{-j\omega(M/2)}
\]

\[
W_{R_2}(e^{j\omega}) = \sqrt{\frac{2}{M}} \frac{\sin(\omega M/4)}{\sin(\omega/2)} e^{-j\omega(M/2)}
\]

\[
W_B(e^{j\omega}) = W_{R_1}(e^{j\omega}) W_{R_2}(e^{j\omega})
\]

\[
= \frac{2}{M} \left( \frac{\sin(\omega M/4)}{\sin(\omega/2)} \right)^2 e^{-j\omega M/2}
\]

Note: The Bartlett window as defined in the text is zero at $n = 0$ and $n = M$. These points are included in the $M + 1$ points.

For $M$ odd, the Bartlett window is the convolution of

\[
r_3[n] = \begin{cases} \sqrt{2/M}, & n = 0, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases} \\
r_4[n] = \begin{cases} \sqrt{2/M}, & n = 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases}
\]

In the frequency domain we have

\[
W_{R_3}(e^{j\omega}) = \sqrt{\frac{2}{M}} \frac{\sin(\omega (M+1)/4)}{\sin(\omega/2)} e^{-j\omega(M/2)}
\]

\[
W_{R_4}(e^{j\omega}) = \sqrt{\frac{2}{M}} \frac{\sin(\omega (M-1)/4)}{\sin(\omega/2)} e^{-j\omega(M/2)}
\]

\[
W_B(e^{j\omega}) = W_{R_3}(e^{j\omega}) W_{R_4}(e^{j\omega})
\]

\[
= \frac{2}{M} \left( \frac{\sin(\omega (M+1)/2)}{\sin(\omega/2)} \right) e^{-j\omega M/2}
\]

(b)

\[
w[n] = \left[ A + B \cos \left( \frac{2\pi n}{M} \right) + C \cos \left( \frac{4\pi n}{M} \right) \right] w_R[n]
\]

\[
W(e^{j\omega}) = \left\{ 2\pi A \delta(\omega) + B \pi \left[ \delta \left( \omega + \frac{2\pi}{M} \right) + \delta \left( \omega - \frac{2\pi}{M} \right) \right] + C \pi \left[ \delta \left( \omega + \frac{4\pi}{M} \right) + \delta \left( \omega - \frac{4\pi}{M} \right) \right] \right\}
\]

where $\otimes$ denotes periodic convolution.

(c) For the Hanning window $A = 0.5$, $B = -0.5$, and $C = 0$.

\[
w_{\text{Hanning}}[n] = \left[ 0.5 - 0.5 \cos \left( \frac{2\pi n}{M} \right) \right] w[n]
\]

\[
W_{\text{Hanning}}(e^{j\omega}) = 0.5W_R(e^{j\omega}) - 0.25W_R(e^{j\omega}) \otimes \left[ \delta \left( \omega + \frac{2\pi}{M} \right) + \delta \left( \omega - \frac{2\pi}{M} \right) \right]
\]

\[
= 0.5W_R(e^{j\omega}) - 0.25 \left[ W_B(e^{j(\omega + \frac{\pi}{M})}) + W_B(e^{j(\omega - \frac{\pi}{M})}) \right]
\]

where

\[
W_B(e^{j\omega}) = \frac{\sin(\omega (M+1)/2)}{\sin(\omega/2)} e^{-j\omega M/2}
\]

Below is a normalized sketch of the magnitude response in dB.

![Normalized Magnitude plot in dB](image-url)