Indirect frequency estimation based on second-order adaptive FIR notch filter

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This paper deals with estimating the frequency of sinusoidal signal buried in broad band noise. The simple unbiased indirect frequency estimation (IFE) algorithm is proposed for a second-order adaptive finite impulse response (FIR) notch filter with constrained zeros. The technique of estimating input noise variance is employed to remove the bias existing in the estimated filter parameter. Also, the difference equation for the convergence of the expectation of the estimated parameter and the closed-form of steady state estimation mean square error (MSE) are derived. In addition, extensive simulations are conducted to corroborate the efficiency of the proposed estimator with the theoretical analysis.

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1. Introduction

It is well known that frequency estimation based on adaptive method plays an important role in radar, sonar, biomedical engineering, control engineering, communication systems, and so on. With the reasons of simplicity and economy, a second-order adaptive notch filter (ANF) together with a simple gradient-based adaptive algorithm is found to be the most popular choice among various methods. From the literature survey, there are two approaches for the ANF, namely, finite impulse response ANF (FIR-ANF) and infinite impulse response ANF (IIR-ANF). The works related the frequency estimation of a sinusoid buried in broad-band noise using the second-order adaptive FIR notch filter are found in [1–3] while the works using IIR-ANF are met in [4–13]. The FIR-ANF is not only simple but also has low complexity, high stability, and fast convergence speed but its disadvantage is wide band-width. The wide band-width characteristic causes distortion in a wide band signal which is passed through a filter and high mean square error (MSE) in the parameter estimate. For the IIR counterpart, although it provides high accuracy of the parameter estimate but has slow convergence speed and its stability becomes the main problem when it is implemented in real-world applications. The slow convergence is due to the flat shape of the error surface whereas the stability problem is due to the finite word-length effects. However, it is difficult to compare performances of FIR and IIR structures because of difference of their structures and characteristics. Consequently, trade-off between obtaining high distorted signal/high MSE of the parameter estimate and slow convergence speed/narrow domain stability is inevitable.

In this paper, an adaptive indirect frequency estimation (IFE) in white noise which is based on adaptive FIR notch filter [3] is presented. The proposed method is based primarily on modifying error surface [1], which is only a function the estimated parameter and its global minimum directly relates the incoming signal frequency. Furthermore, the bias removal technique developed by Punchalard et al. [3] is also applied. Consequently, simple unbiased IFE algorithm is achieved. In addition, the analysis methods adopted by So and Ching [2] and Punchalard [3] have been applied to analyze the statistical

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performance of the proposed estimator. The difference equation for the convergence of the expectation of the parameter estimate and the closed-form expression of steady state estimation MSE are also investigated. Finally, the simulation results are conducted to confirm with the theoretical analysis and to demonstrate the performance of the proposed estimator.

2. Adaptive FIR notch filter and the IFE

2.1. Adaptive FIR notch filter

In this section, the second-order adaptive FIR notch filter is addressed. The transfer function of this system is based on [3] and is given by

\[ H(z) = 1 + az^{-1} + z^{-2}. \]  

(1)

As shown in Eq. (1), \( a \) is a filter coefficient whose actual value \( a_0 \) is equal to \(-2 \cos \omega_0\). In addition, \( \omega_0 \) is the frequency of the incoming sinusoid buried in noise which is

\[ x(k) = A \cos(\omega_0 k + \theta) + v(k), \]  

(2)

where \( A \) is the signal amplitude, \( \theta \) is the signal phase with uniform distribution over \([0, 2\pi]\), and \( v(k) \) is assumed to be a statistically independent additive white Gaussian noise with zero-mean and variance \( \sigma_v^2 \). Referring to Eq. (1), the output or the error signal of that filter can be expressed as

\[ e(k) = x(k) + ax(k - 1) + x(k - 2). \]  

(3)

The involved parameter \( a \) is adjusted so as to minimize the mean squared value of the error signal shown in Eq. (3).

2.2. IFE

The IFE to adjust the filter parameter \( a \) is discussed in this section. The error function of the proposed filter can be expressed as

\[ J(a) = E[e^2(k)] = \frac{1}{2\pi} \int_0^{2\pi} |H(e^{io})|^2 S_a(e^{io}) \, d\omega, \]  

(4)

where \( E[\cdot] \), \( |H(e^{io})|^2 \) and \( S_a(e^{io}) \) are expectation operator, magnitude squared of the system frequency response and power spectral density of an incoming signal, respectively. The parameters \( |H(e^{io})|^2 \) and \( S_a(e^{io}) \) are, respectively, defined by

\[ |H(e^{io})|^2 = (a + 2 \cos \omega)^2 \]  

(5)

and

\[ S_a(e^{io}) = \frac{A^2}{4} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \sigma_v^2, \]  

(6)

where \( \delta(\cdot) \) is the Dirac delta function. By substituting Eqs. (5) and (6) into Eq. (4), it results in

\[ J(a) = \frac{A^2}{2} (a + 2 \cos \omega_0)^2 + \sigma_v^2(2 + a^2). \]  

(7)

As can be seen in Eq. (7), the second term on the right-hand side (RHS), which is the noise term, is a function of \( a \). A global minimum of \( J(a) \), rather than at \( a = -2 \cos \omega_0 \), then occurs at \( a = -(A^2/(A^2 + 2\sigma_v^2)) \cos \omega_0 \) which is known to be bias. To overcome this problem, both sides of Eq. (7) are subtracted by \( \sigma_v^2(2 + a^2) \) which yields [1]

\[ J_1(a) = J(a) - \sigma_v^2(2 + a^2). \]  

(8)

By using this method, the minimum point is ensured to be at \( a = -2 \cos \omega_0 \).

Now, let us introduce the adaptive algorithm. The IFE which is used to update the parameter \( a \) is defined by

\[ a(k + 1) = a(k) - \frac{\mu}{2} \frac{\partial J_1(a(k))}{\partial a(k)}, \]  

(9)

where \( \mu \) is the step size parameter controlling the speed of convergence and is positive real number by definition, \( a(k) \) is the estimate of \( a \) at time \( k \),

\[ J_1(a(k)) = \frac{e^2(k) - \sigma_v^2(k)(2 + a^2(k))}{2}, \]  

(10)

is the estimate of \( J_1(a) \). From Eq. (10), \( \sigma_v^2(k) \) is the estimate of input noise variance at time \( k \). By substituting Eq. (10) into Eq. (9), it results in

\[ a(k + 1) = a(k) - \mu(e(k)g(k) - \sigma_v^2(k)a(k)), \]  

(11)

where \( g(k) \) is the gradient signal which is given by

\[ g(k) = \frac{\partial e(k)}{\partial a(k)} = x(k - 1), \]  

(12)

and is generated by the gradient filter \( G(z) \) which is fed by \( x(k) \). The gradient filter transfer function is given by

\[ G(z) = z^{-1}. \]  

(13)

It is observed from Eq. (11) that the estimate of noise variance \( \sigma_v^2(k) \) is unknown, implying that this algorithm is not practical if the noise variance is not available. To overcome this problem, the technique of estimating \( \sigma_v^2(k) \) adopted in [3] is employed, which is

\[ \sigma_v^2(k) \approx x(k)e(k). \]  

(14)

Eq. (14) is the estimate of the input noise variance at time \( k \). By using Eq. (14) into Eq. (11), it then yields

\[ a(k + 1) = a(k) - \mu(e(k)g(k) - \sigma_v^2(k)x(k)). \]  

(15)

It is seen that the proposed algorithm Eq. (15) is similar to those of adaptive algorithms adopted in [1–3], but the IFE technique is employed instead. The following conclusive remarks of those three methods are drawn as follows:

R1: In [1], the input noise variance must be known \( a \) priori.

R2: In [2], the employed technique is different than that of the proposed method.

R3: In [3], the analysis result for steady state estimation MSE is not yet precise.

To reduce computation for the adaptive algorithms given in [1–3], the value of the cosine and sine functions are retrieved by look-up table from the pre-stored \( L \)-length cosine and sine vectors of the forms \([1, \cos(\pi/L) \ldots \cos(\pi(L - 1)/L)] \) and \([0, \sin(\pi/L) \ldots \sin(\pi(L - 1)/L)] \). It is noted that when \( L \) increases, the frequency resolution increases but large memory is needed. Thus, when
comparing in terms of the memory requirement, the proposed technique is preferable because any look-up tables are not required.

In the next section, the performance analysis of the proposed unbiased gradient-based adaptive algorithm Eq. (15) is addressed.

3. Performance analysis

In this section, the steady state analysis approach of [2] is adopted in the proposed technique where the details are given as follows.

3.1. Steady state error and gradient signals

Assuming that the filter parameter \( a(k) \) approaches its actual value \( a_0 = -2 \cos \omega_0 \) at steady state. The transfer function \( H(z) \) Eq. (1) is therefore expressed as

\[
H(e^{j\omega_0}) = \delta_0(k) e^{-j\omega_0},
\]

(16)

where \( \delta_0(k) = a(k) - a_0 \) is the estimation error. Without taking consideration of \( \delta_0(k) \) of Eq. (16), the magnitude and phase response are, respectively, given by

\[
B = |e^{-j\omega_0}| = 1
\]

(17)

and

\[
\angle H(e^{j\omega_0}) = -\omega_0.
\]

Thus, the steady state error signal can be shown as

\[
e_s(k) = A\delta_0(k) \cos(\omega_0 k + \theta - \omega_0) + v_1(k),
\]

(19)

where subscript \( s \) stands for steady state and \( v_1(k) \) is assumed as a zero-mean Gaussian noise at the output of \( H(z) \) excited by \( v(k) \). The variance of \( v_1(k) \) can be computed by using the Parseval relation [14] which is

\[
\sigma^2_{v_1} = \sigma^2_v \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega = 2(1 + 2 \cos^2 \omega_0) \sigma^2_v.
\]

(20)

Similarly, at steady state, Eq. (13) can be expressed as

\[
G(e^{j\omega_0}) = e^{-j\omega_0}.
\]

(21)

It is seen that Eq. (21) has unit magnitude \( |G(e^{j\omega_0})| = 1 \) and phase \( \angle G(e^{j\omega_0}) = \text{Eq. (18)} \). Consequently, the steady state gradient signal is therefore given by

\[
g_r(k) = A \cos(\omega_0 k + \theta - \omega_0) + v_2(k),
\]

(22)

where \( v_2(k) \) is assumed to be a zero-mean Gaussian noise at the output of \( G(z) \) excited by \( v(k) \). The variance of \( v_2(k) \) is given by

\[
\sigma^2_{v_2} = \sigma^2_v \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega = \sigma^2_v.
\]

(23)

It is noted that the steady state error and gradient signals derived above are used to analyze the performance at steady state of the proposed algorithm. Although the analysis technique presented in this work follows the theoretical framework in [2] which is adopted in this work but their analyses are different in details.

3.2. Convergence behavior

By using Eqs. (2), (19), and (22) into Eq. (15) and taking expectation of the result where it is assumed that the fluctuations in the estimation error \( \delta_0(k) \) are small such that the coefficient \( a(k) \) involved in the last term on the RHS of Eq. (15) can be replaced by its mean value \( a_0 \), which then yields

\[
E[a(k + 1)] = (1 + \mu \psi_1) E[a(k)] + \mu \psi_2,
\]

(24)

where

\[
\psi_1 = \frac{1}{2} A^2 (a_0 \cos \omega_0 - 1),
\]

\[
\psi_2 = -a_0 \psi_1.
\]

In addition, the relation which is \( E[a(k + 1)]_{k \to \infty} = E[a(k)]_{k \to \infty} = E[a(\infty)] \) is substituted into Eq. (24), then

\[
E[a(\infty)] = a_0.
\]

(25)

The expression shown in Eq. (24) is the difference equation for the convergence of the expectation of the parameter estimate which can be used to study the convergence behavior of the estimated parameter. Eq. (25) is the mean value of the estimated parameter at steady state. It is obvious from Eq. (25) that the mean value of the parameter estimate equals its true value \( a_0 \), implying that the proposed algorithm is unbiased.

Note that, the derivation of Eq. (24) is obtained by replacing \( \delta_0(k) \) in Eq. (19) with \( a(k) - a_0 \) and assuming that the cosine waves and the noise components \( v(k) \), \( v_1(k) \), and \( v_2(k) \) are uncorrelated with each other. Moreover, the terms \( E[v_1(k) v_2(k)] = R_{12} = a_0 \sigma^2_v \) and \( E[v(k) v_1(k)] = \sigma^2_v \) [3] are also employed.

3.3. MSE analysis

In this section, the MSE analysis of the proposed technique is discussed. The analysis begins with subtracting \( a_0 \) from both sides of Eq. (15), then squaring, arranging, and averaging the result which yields

\[
2E[\delta_0(k) e(k) (g(k) - a_0 x(k))] = \mu E[e^2(k) (g(k) - a_0 x(k))^2].
\]

(26)

Note that, the coefficient \( a(k) \) involved in Eq. (26) is replaced with \( a_0 \) for analytical simplicity. In addition, the component related to signal only in the RHS of Eq. (26) can be approximated as (see Appendix A for its derivation)

\[
\mu E[e^2(k) (g(k) - a_0 x(k))^2] \text{ due to signal only} = \mu \eta_1 E[a^2_0(k)],
\]

(27)

where

\[
\eta_1 = \frac{A^4}{4} \left( \frac{3}{2} - 3a_0 \cos \omega_0 + \frac{1}{2} a^2_0 [2 + \cos 2 \omega_0] \right).
\]

Furthermore, the component related to noise only in the RHS of Eq. (26) can be estimated by (see Appendix A for its derivation)

\[
\mu E[e^2(k) (g(k) - a_0 x(k))^2] \text{ due to noise only} = \mu \eta_2.
\]

(28)
where
\[ \eta_2 = 2\sigma_y^2(1 + a_0^2)(2\cos^2\omega_0 + 1). \]

Using the method in [2], the quantity in the left-hand side (LHS) of Eq. (26) can be rewritten as
\[
2E[\delta_2(k)|g(k) - a_0x(k)|] = 2E[\delta_2(k - 2)|g(k) - a_0x(k)|] - 2\mu E \left[ \sum_{i=1}^{2} (e(k - i)|g(k - i) - a_0x(k - i)) \right] \\
\times (e(k)|g(k) - a_0x(k)|) \approx 2E[\delta_2(k)|e(k)|g(k) - a_0x(k)|] - 2\mu E \left[ \sum_{i=1}^{2} (e(k - i)|g(k - i) - a_0x(k - i)) \right] \\
\times (e(k)|g(k) - a_0x(k)|). \tag{29}
\]
The first term in the RHS of Eq. (29) related to both signal and noise can be approximated as (see Appendix B for its derivation)
\[
2E[\delta_2(k)|e(k)|g(k) - a_0x(k)|] = 2\eta_0 E[\delta_2^2(k)], \tag{30}
\]
where
\[ \eta_0 = \frac{1}{2}A^2(1 - a_0 \cos \omega). \]

Similarly, the second term can be evaluated as (see Appendix C for its derivation)
\[
2\mu \left[ \sum_{i=1}^{2} (e(k - i)|g(k - i) - a_0x(k - i)) (e(k)|g(k) - a_0x(k)|) \right] \\
= 2\mu (\eta_3 + \eta_5) E[\delta_2^2(k)] + \eta_4. \tag{31}
\]
where \( x(k - i), g(k - i), \) and \( e(k - i), i = 1, 2 \) are, respectively, defined by
\[
x(k - i) = A \cos(\omega_0 k + \theta - i\omega_0) + v(k - i), \tag{32}
g(k - i) = A \cos(\omega_0 k + \theta - (i + 1)\omega_0) + v_2(k - i), \tag{33}
e(k - i) \approx A \delta_2(k) \cos(\omega_0 k + \theta - (i + 1)\omega_0) + v_1(k - i) \tag{34}
\]
and
\[
\eta_3 = \frac{A^4}{4} \left[ \frac{1}{2} [2 + \cos 2\omega_0] - \frac{3}{2} a_0^2 \cos \omega_0 - a_0 \left[ \cos \omega_0 + \frac{1}{2} \cos 3\omega_0 \right] \right],
\]
\[
\eta_4 = (1 - 3a_0^2) \sigma_v^2,
\]
\[
\eta_5 = \frac{A^4}{4} \left[ \begin{array}{c}
\frac{1}{2} [2 + \cos 2\omega_0] \\
- a_0^2 \cos \omega_0 + a_0^2 \cos 3\omega_0 \\
- a_0 \cos \omega_0 + a_0 \cos 5\omega_0 \\
+ a_0^2 \cos 2\omega_0 + a_0^2 \cos 2\omega_0 \end{array} \right].
\]
Substituting Eqs. (27)–(31) into Eq. (26) and setting \( k \to \infty \), it provides
\[
E[\delta_2^2(\infty)] = \frac{\mu (\eta_2 + 2\eta_4)}{2\eta_0 - \mu (\eta_1 + 2\eta_3 + 2\eta_5)}. \tag{35}
\]
Eq. (35) is the closed-form expression for steady state estimation MSE of the proposed IFE for a second-order adaptive FIR notch filter. It is observed that the MSE directly depends on the step size parameter \( \mu \) and input noise variance \( \sigma_v^2 \). Namely, the larger the step size \( \mu \) or \( \sigma_v^2 \) is, the higher the MSE will be. This means that in the noise-free case, the estimation MSE will vanish. However, in a practical point of view, the value of \( \mu \) cannot be zero. Thus, the estimation MSE of the proposed algorithm will not vanish even if \( \sigma_v^2 = 0 \). Therefore, when the SNR value is high, the estimation MSE of the proposed algorithm depends only on the step size parameter \( \mu \) and will not vanish. Note that, the derivations of Eqs. (27)–(31) are obtained by using the following assumptions:

A1: The signals and noise components are uncorrelated with each other.
A2: The cosine waves have zero-mean and variance of 0.5.
A3: All noise components are zero-mean and Gaussian distribution.
A4: The method of Gaussian moment factoring theorem is employed under the assumption that \( v(k), v_1(k), v_2(k), v(k - i), v_1(k - i), \) and \( v_2(k - i), i = 1, 2 \) are jointly Gaussian distribution.

### 3.4. Stability analysis

Stability of the adaptive algorithm shown in Eq. (15) is obtained by using the appropriated value of the step size parameter \( \mu \). The analytical bound of \( \mu \) is thus discussed. First, Eq. (35) is focused on. Because the estimation MSE is always positive, hence, from its numerator and denominator, it is found that
\[
\mu > 0 \tag{36}
\]
and
\[
\mu \leq \frac{2\eta_0}{\eta_1 + 2(\eta_3 + \eta_5)}. \tag{37}
\]
respectively. Then, the bound of step size is
\[
0 < \mu < \frac{2\eta_0}{\eta_1 + 2(\eta_3 + \eta_5)}. \tag{38}
\]

### 4. Results and discussions

To evaluate the sinusoidal frequency estimation performance of the IFE in the presence of additive white Gaussian noise for non-stationary conditions, the computer simulations are tested. The obtained results are compared with three gradient-based adaptive algorithms which are claimed to provide unbiased frequency estimate, namely, the direct frequency estimation algorithm 1 (DFE1) [1], the direct frequency estimation algorithm 2 (DFE2) [2], and the direct frequency estimation algorithm 3 (DFE3) [3]. In the comparison of the calculation requirement of all algorithms, the computational complexity per iteration of each algorithm is concluded in Table 1.
As seen in Table 1, the proposed IFE requires only four additions or subtractions and four multiplications, where any look-up tables are not required. The length-$L$ sine and cosine vectors are assigned to be 1000. Thus, the resolution of $\pi/1000$ rad/sample is obtained. Other chosen parameters are indicated in the captions of the figures.

**Example 1.** The step changing frequency sinusoidal signal is applied to the tested filters. During the first 15000 samples, an incoming signal has the frequency of $0.9\pi$ rad/sample and is suddenly changed to $0.4\pi$ rad/sample at the 15000th sample. The last 1000 samples of 100 independent runs of the simulation are ensemble averaged to obtain their mean values. The comparison between theory and simulation of the IFE and the simulated results of all algorithms are shown in Figs. 1 and 2, respectively. In Fig. 1, although the theoretical and simulated results show some difference at the transient state, but they fit well at the steady state region. In Fig. 2, it is observed that at a given step parameter value $\mu$, the convergence property of the three compared algorithms is better than that of the IFE. But the IFE is marginally better than others in computation.

**Example 2.** Comparisons between theoretical and simulated MSEs of all algorithms with respect to step size $\mu$, signal to noise ratio (SNR), and signal frequency $\omega_0$ are demonstrated in Figs. 3–5, respectively. In Fig. 3, the theoretical and simulated MSE of the IFE show good agreement, particularly when $\mu < 1 \times 10^{-3}$. Furthermore, the larger the step size is, the higher the MSE will be. It is also seen that the DFE1 provides higher MSE value than those of others whereas the DFE2, the DFE3, and the IFE provide the same MSE. In Fig. 4, analytical and measured MSE of the IFE are in accordance when SNR < 25 dB but when SNR > 25 dB, large error exists. Also, the DFE1 provides the highest MSE whereas the DFE2, the DFE3, and the IFE provide no difference in their MSE values. In addition, the larger the SNR is, the smaller the MSE will be. As has been mentioned in the end of Section 3.3, the value of the practical MSE will not vanish even if $\sigma^2 = 0$ or SNR = $\infty$. This is because of the effect of step size parameter $\mu$. As a result, the deviation between analytical
D4: The DFE1, the DFE3, and the IFE provide the same MSE value at \( \omega_0 \approx 0 \) and \( \pi \).

D5: When \( \omega_0 \in [0.15\pi, 0.85\pi] \), the MSE obtained from the DFE2 is improved.

Since the choice of initial condition has a major impact on the convergence behavior of the proposed algorithm (due to the characteristic of the error function Eq. (8)), then the following initial condition must be satisfied:

\[
-1 < a(0) < 1, \quad \forall \omega_0 \in [0, \pi],
\]

where \( a(0) \) is the estimate of \( a \) evaluated at \( k = 0 \). The derivation of the initial condition Eq. (39) is obtained by taking the first and second derivatives of Eq. (8).

5. Conclusion

This work presents a gradient-based adaptive algorithm for a second-order adaptive FIR notch filter. It is shown that the proposed technique can remove the bias existing in the parameter estimate, therefore, the knowledge of input noise variance is not required. In addition, the mathematical analysis of the difference equation for the convergence in the expectation of the estimated parameter and the steady state estimation MSE are provided. The obtained simulation results show that the proposed frequency estimator provides low MSE when the signal frequency is in the vicinity of 0 and \( \pi \) rad/sample. Since the proposed IFE is marginal better than the compared algorithms in computation at the cost of some degradation in convergence, the convergence property of the IFE must be further improved.

Appendix A. The derivations of Eqs. (27) and (28)

The term on the RHS of Eq. (27) can be rewritten as

\[
\mu E[e_t^2(k)|g_s(k) - a_0x(k)]
= \mu \left( E[e_t^2(k)g_s^2(k)] - 2a_0E[x(k)g_s(k)e_t^2(k)] + a_0^2E[x^2(k)e_t^2(k)] \right)
= \mu(M_1(k) - 2a_0M_2(k) + a_0^2M_3(k)),
\]

where

\[
M_1(k) = E[e_t^2(k)g_s^2(k)],
M_2(k) = E[x(k)g_s(k)e_t^2(k)],
M_3(k) = E[x^2(k)e_t^2(k)].
\]

The results of computing Eqs. (41)–(43) due to the signal only can be expressed as

\[
M_1(k)_{\text{due to signal}} = A^4B^4E[\delta_0^2(k)\cos^4(\omega_0k + \theta - \omega_0)]
= A^4B^4\cos^4(\omega_0k + \theta - \omega_0)E[\delta_0^2(k)]
= \frac{2}{3}A^4B^4E[\delta_0^2(k)],
\]
Similarly, the components due to noise only can be evaluated as

\[ M_4(k)_{\text{due to noise}} = E[\delta_0(k)v_1(k)v_2(k)] = E[\delta_0(k)]E[v_1(k)v_2(k)] = E[\delta_0(k)]R_{12}. \]  

\[ M_5(k)_{\text{due to noise}} = E[\delta_0(k)v_1(k)] = E[\delta_0(k)]E[v_1(k)] = E[\delta_0(k)]\sigma_1. \]  

After substituting Eqs. (53)–(56) into Eq. (50) we obtain Eq. (30).

**Appendix C. The derivation of Eq. (31)**

The term on the LHS of Eq. (31) without the constant 2\mu can be rewritten as

\[
E \left[ \sum_{i=1}^{2} [e_i(k)(g_i(k) - 2a_0x_i(k))] \right]
\]

\[
= \begin{bmatrix}
    -a_0e_i(k)g_i(k)g_i(k-1)a_0x_i(k-1) \\
    -a_0e_i(k)a_0x_i(k)g_i(k-1) \\
    +a_0^2x_i(k)e_i(k-1)e_i(k-1) \\
    +e_i(k)g_i(k)e_i(k-2) \\
    -a_0e_i(k)e_i(k-1)e_i(k-2) \\
    -a_0x_i(k)e_i(k-1)e_i(k-2) \\
    +a_0^2x_i(k)e_i(k-2)e_i(k-2)
\end{bmatrix}
\]

\[
= \sum_{i=1}^{2} [M_{6,i}(k) - a_0M_{7,i}(k) - a_0M_{8,i}(k) + a_0^2M_{9,i}(k)].
\]  

where

\[ M_{6,i}(k) = E[e_i(k)g_i(k)e_i(k-1)a_0x_i(k-1)], \]

\[ M_{7,i}(k) = E[e_i(k)a_0x_i(k)e_i(k-1)g_i(k-1)], \]

\[ M_{8,i}(k) = E[x_i(k)e_i(k-1)e_i(k-2)g_i(k-1)]. \]

\[ M_{9,i}(k) = E[x_i(k)e_i(k-1)e_i(k-2)x_i(k-1)]. \]

Applying the signals into Eqs. (58)–(61) for \( i = 1, 2 \) results in

\[ M_{6,i}(k)_{\text{due to signal}} = A^4B^4 \left\{ \begin{array}{l}
    \frac{1}{2}A^4B^4(2 + \cos(2a_0k))E[\delta_0^2(k)], \\
    i = 1,
\end{array} \right. \]

\[ M_{7,i}(k)_{\text{due to signal}} = A^4B^4 \left\{ \begin{array}{l}
    \frac{1}{2}A^4B^4(2 + \cos(4a_0k))E[\delta_0^2(k)], \\
    i = 2.
\end{array} \right. \]
\[ M_{7,j}(k) \text{ due to signal} = A^4 \cos(\omega_0 k + \theta) \cos(\omega_0 k + \theta - (i + 1)\omega_0) \]
\[ = A^4 \cos(\omega_0 k + \theta) \cos(\omega_0 k + \theta - (i + 1)\omega_0) \]
\[ = \left\{ \begin{array}{ll}
\frac{1}{2} A^4 \cos(\omega_0) & i = 1, \\
\frac{1}{2} A^4 \cos(\omega_0) + \frac{1}{2} \cos(3\omega_0) \delta^2(k), & i = 2.
\end{array} \right. \] (63)

\[ M_{8,j}(k) \text{ due to signal} = A^4 \cos(\omega_0 k + \theta) \cos^2(\omega_0 k + \theta - (i + 1)\omega_0) \]
\[ = A^4 \cos(\omega_0 k + \theta) \cos^2(\omega_0 k + \theta - (i + 1)\omega_0) \]
\[ = \left\{ \begin{array}{ll}
\frac{1}{4} A^4 \cos(\omega_0) & i = 1, \\
\frac{1}{4} A^4 \cos^2(\omega_0) + \frac{1}{4} \cos(5\omega_0) \delta^2(k), & i = 2.
\end{array} \right. \] (64)

\[ M_{9,j}(k) \text{ due to noise} = A^4 \cos(\omega_0 k + \theta) \cos(\omega_0 k + \theta - (i + 1)\omega_0) \]
\[ = A^4 \cos(\omega_0 k + \theta) \cos(\omega_0 k + \theta - (i + 1)\omega_0) \]
\[ = \left\{ \begin{array}{ll}
\frac{1}{4} A^4 \cos(2\omega_0) \delta^2(k), & i = 1, \\
\frac{1}{4} A^4 \cos^2(2\omega_0) + \frac{1}{4} \cos(2\omega_0) \delta^2(k), & i = 2.
\end{array} \right. \] (65)

Furthermore, applying the noises into Eqs. (58)-(61) for \( i = 1, 2 \) results in

\[ M_{8,j}(k) \text{ due to noise} = E[v_1(k)v_2(k)]v_1(k - i)v_2(k - i) \]
\[ = E[v_1(k)v_2(k)]E[v_1(k - i)v_2(k - i)] + E[v_1(k)]v_1(k - i)E[v_2(k)v_2(k - i)] + E[v_1(k)v_1(k - i)]E[v_2(k)v_2(k - i)] \]
\[ = \left\{ \begin{array}{ll}
R_{12}^2 + \sigma_v^4, & i = 1, \\
R_{12}^2, & i = 2.
\end{array} \right. \] (66)

\[ M_{7,j}(k) \text{ due to noise} = E[v_1(k)v_2(k)]v_1(k - i)v_2(k - i) \]
\[ = E[v_1(k)v_2(k)]E[v_1(k - i)v_2(k - i)] + E[v_1(k)]v_1(k - i)E[v_2(k)v_2(k - i)] + E[v_1(k)v_1(k - i)]E[v_2(k)v_2(k - i)] \]
\[ = \left\{ \begin{array}{ll}
2\sigma_v^4 + 2R_{12}^2 \sigma_v^2, & i = 1, \\
\sigma_v^4 R_{12}, & i = 2.
\end{array} \right. \] (67)

After substituting Eqs. (62)-(69) into Eq. (57) we obtain Eq. (31).

References