We have experience in finding roots of polynomials that are given in a 'nice' form, or for quadratic and linear polynomials. However, what about finding roots for a polynomial like 
\[ P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90 \]

As of now, all of our processes involve setting \( P(x) = 0 \) and then finding \( x \). And that would work here, but we do not have the any formulas to solve directly for \( x \) values like we do in the quadratic case. There is a formula for a quartic equation, however, it is very complicated compared to the quadratic equation.

In general, polynomial equations higher than fourth degree are incapable of algebraic solution in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions.

So to find the roots, we can use other methods, such as approximation numerically.

One such way is to simply graph the polynomial and visually inspect the graph for the roots. Even within this graphical approach it might be nice to have an idea of the upper and lower bounds for the real zeros of polynomials.

However, there are more accurate ways of approximating roots, one such way involves using synthetic division.

**Theorem 1: Upper and Lower Bound Theorem**

Let \( P(x) \) be a polynomial of degree \( n > 0 \) with real coefficients, \( a_n > 0 \):

1. **Upper bound**: A number \( r > 0 \) is an upper bound for the real zeros of \( P(x) \) if, when \( P(x) \) is divided by \((x - r)\) by synthetic division, all numbers in the quotient row, including the remainder, are nonnegative.

2. **Lower bound**: A number \( r < 0 \) is a lower bound for all the real zeros of \( P(x) \) if when \( P(x) \) is divided by \((x - r)\) by synthetic division, all numbers in the quotient row, including the remainder, alternate in sign.

As a reminder, the Quotient row, is the row under the sign in synthetic division (it contains the coefficients for your answer \( Q(x) \))

\[
\begin{array}{c|cccc}
  & d & \cdots & a_n & r \\
\hline
q & q_1 & q_2 & \cdots & q_n R & Quotient row
\end{array}
\]

So, this tells us that, with synthetic division, you are able to locate upper and lower bounds for the real zeros of the polynomials without the aid of graphing.

We will work through one example of finding bounds, this way, although it is not going to be a focus of this
Example 2 (pg 279):

\[ P(x) = x^3 - 30x^2 + 275x - 720 \]

Find the smallest possible integer multiple of 10 and the largest possible negative integer multiple of 10, that by the Upper and Lower Bound theorem, are the Upper Bound (UB) and Lower Bound (LB) of the polynomial's real zeros.

The first synthetic division, would be for the number 10. (We would want the quotient row to have all positive signs to indicate that the number chosen is an Upper Bound). If all the signs are positive, we stop, if not, choose 20, and 30 and so on until all the signs in the quotient row are positive.

For the LB, we would choose our first number to be -10, since that is the largest negative integer multiple of 10. Now here we want the quotient row to alternate signs, then we are done and have found the LB on the real zeros.

With the bounds in place, we can then move onto using the location theorem and the bisection method to locate the actual zeros, and not just the bounds on them.

Consider this. Say we have some polynomial, \( P(x) \), and we evaluate it at a number, call it \( a \). We find that \( P(a) \) is -2. Now if we evaluate the same polynomial at a different number, call it \( b \), and we find that \( P(b) \) is 4, there is something that we can say about the polynomial between the values \( a \) and \( b \) that we chose.
Theorem 2: Location Theorem
Suppose that a function $f$ is continuous on an interval $I$ that contains numbers $a$ and $b$. If $f(a)$ and $f(b)$ have opposite signs, then the graph of $f$ has at least one $x$-intercept between $a$ and $b$.

See the handout for work on using the location theorem and the bisection method.