AN IMPECUNIOUS GAMBLER
IN A KIND-HEARTED CASINO

by

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Technical Report 86-1

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January, 1986

QUIT WHILE YOU ARE AHEAD

or

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ABSTRACT

Gamblers, and other people who are about to lose, are often advised to "Quit while you are ahead." In practice this advice means, "Don't start," or, "Don't play for long," or, "Quit if you ever get ahead." Another, rather fanciful, interpretation is that a gambler should quit just before he starts losing. Of course, it is impossible to use such a strategy, since the decision to stop would be based on the future, but it is amusing to consider what would happen if such a strategy were possible.

Imagine a gambler who starts play with a fortune of \$1. He plays a game, winning or losing \$1 on each play until he goes broke. When this happens he laments not having quit "while he was ahead." When the casino offers to return the \$1 the gambler has lost he replies, sorrowfully: "But I was once as much as \$$X$ ahead." The casino responds: "We don't want any dissatisfied customers, so we will return your \$$X$ as well." The question is, how much could a gambler expect to win in such a kind-hearted casino?

This problem bears a close resemblance to the famous Petersburg problem and leads to some amusing mathematical conclusions. While the problem is obviously fanciful, a number of biologists seem unwittingly to have made equally fanciful assumptions about how animals should distribute themselves in a patchy environment. I mention several examples of such biological work and point out that their arguments are invalid. It is interesting to note that the distribution of "winnings" of a gambler in a kind-hearted casino is very similar to the distribution of the number of papers written by individual

scientists (Lotka's law) and to the distribution of the number of times that individual scientific papers are cited in the literature.

AN IMPECUNIOUS GAMBLER IN A KIND-HEARTED CASINO

Imagine a gambler who starts play with $1 (an "impecunious gambler") and plays repeatedly, winning $1 with probability p, and losing $1 with probability $q = 1 - p$ on each play. The gambler continues to play until he loses his $1. Then the casino returns his $1 plus the largest amount the gambler was ever ahead, $\$X$. I refer to a casino willing to do this as a "kind-hearted casino." The question is: What is the expected value of X? And, more generally: What is the distribution of X?

If we denote the probability that $X = x$ by $P(X = x)$, we can write

$$EX = \sum_{x=1}^{\infty} xP(X = x), \qquad \text{or}$$

$$EX = \sum_{x=1}^{\infty} P(X \geq x). \tag{1}$$

Values of $P(X \geq x)$ are found by considering an absorbing random walk (the "gambler's ruin" problem), starting at integer y, where $0 \leq y \leq n$. Steps are taken at random, up with probability p, down with probability $q = 1 - p$, until the process reaches 0 or n. If, for a walk starting at y, we indicate the probability that n is hit before 0 by the expression, $h(y, n)$, then we have

$$h(y, n) = qh(y - 1, n) + ph(y + 1, n)$$

which is a difference equation which can be written as

$$ph(y + 2, n) - h(y + 1, n) + qh(y, n) = 0. \tag{2}$$

Solving the auxillary equation

$$ph^2 - h + q = 0, \qquad \text{or}$$
$$(ph - q)(h - 1) = 0 \tag{3}$$

yields solutions, $h = q/p$ and $h = 1$. Thus, we have

$$h(y,n) = a(q/p)^y + b \qquad \text{if } p \neq q, \text{ and} \tag{4}$$

$$h(y,n) = ay + b \qquad \text{if } p = q \tag{5}$$

We have side conditions

$$h(0,n) = 0, \qquad \text{and} \tag{6}$$

$$h(n,n) = 1. \tag{7}$$

Solving for a and b, we have

$$h(y,n) = \frac{(q/p)^y - 1}{(q/p)^n - 1} \qquad \text{if } p \neq q, \text{ and} \tag{8}$$

$$h(y,n) = \frac{y}{n} \tag{9}$$

if $p = q$.

To find $P(X \geq x)$, set $y = 1$ and $n = x + 1$, in (8) or (9). We have

$$P(X \geq x) = h(1, x + 1). \tag{10}$$

Using (1), (10) and (8) or (9), we can calculate the expected winnings of an impecunious gambler in a kind-hearted casino.

SOME RESULTS

FAIR GAME, UNLIMITED PAYOFF.

For a fair game $(p = q)$, we have

$$P(X \geq x) = \frac{1}{(x+1)}, \tag{11}$$

and

$$EX = \sum_{x=1}^{\infty} \frac{1}{(x+1)} = \infty. \tag{12}$$

In other words, the expectation does not exist if the game is fair.

FAIR GAME, LIMITED PAYOFF.

Suppose that the casino is willing to return to the gambler an amount equal to the maximum amount he is ever ahead, as long as this amount does not exceed some limit, $\$T$. If the gambler is sometime more than $\$T$ ahead, the casino will return only $\$T$. In this case, the expected winnings are

$$EX = \sum_{x=1}^{T} \frac{1}{(x+1)} \sim \ln T. \tag{13}$$

Exact values for expected winnings (rounded to the nearest cent), for some possible limits, are

Limit $= T$	EX
10	\$2.02
100	\$4.20
1000	\$6.49
10000	\$8.79

Notice that if the payoff is limited, then the expected payoff is relatively low, even for a high limit.

AN UNFAIR GAME.

Suppose that our gambler plays a game which is slightly unfavorable, such as craps. If he bets "pass," a pair of dice are tossed, and he wins if a seven or eleven occurs and loses if two, three or twelve occurs. If four, five, six, eight, nine or ten occurs, the dice are tossed until either that number recurs or a seven is tossed. The gambler then wins if his number is tossed before a seven, and he loses if a seven is tossed before his number. The probability of winning such a bet is $p = 244/495$. The odds against winning such a bet are $q/p = 251/244$. Using (1), (8) and (10) we find that, for a gambler playing craps in a kind-hearted casino, the expected payoff is \$3.21. For roulette, the odds against winning each bet are 20/18, and the average payoff in a kind-hearted casino is \$2.01.

One can calculate the expected payoff for our gambler playing games with other odds. If \$1 is won or lost on each play, the expected loss on each play is $q - p$, which is \$.01414 for craps and \$.05263 for roulette. For some other values we have

$q - p$	EX
.10	\$1.47
.05	\$2.05
.01	\$3.54
.005	\$4.21
.001	\$5.80
.0005	\$6.49

For games that are only slightly unfair the expected gain in a kind-hearted casino is roughly proportional to $-\ln(q - p)$.

If a gambler started playing with a fortune of $\$F$, greater than \$1, and played for \$1 on each play until he went broke, then getting back his original stake plus $\$X$, the maximum amount he was ever ahead, he would expect to gain more than if he started with only \$1. In this case, instead of expression (10), we would have

$$P(X \geq x) = h(F, x + F). \tag{14}$$

For some initial fortunes, the expected payoff to a gambler playing craps would be

F	EX
1	\$3.21
2	\$5.21
3	\$7.38
4	\$8.99
5	\$10.40
10	\$15.70
20	\$22.08
50	\$30.16
100	\$33.79
200	\$34.80
400	\$34.86

As a gambler's initial fortune increases the expected payoff increases, but at a decreasing rate that approaches zero, which happens for initial fortunes of about \$400 if the gambler plays craps.

A GENERALIZED LAW OF LARGE NUMBERS FOR AN IMPECUNIOUS GAMBLER PLAYING A FAIR GAME IN A KIND-HEARTED CASINO

For a fair game ($p = q$) and a gambler starting with \$1, the probability distribution of the payoff, X, is specified by (11). From (12) we saw that the expectation of X does not exist (is infinite). The same thing is true of the famous Petersburg problem (Feller, 1968), in which a gambler tosses a fair coin until "heads" occurs. If this takes Y tosses, the gambler wins $\$2^Y$, which has probability 2^{-Y} . Thus, the gambler's expected payoff is

$$EX = \sum_{y=1}^{\infty} 2^y 2^{-y} = 1 + 1 + 1 + \ldots = \infty. \tag{15}$$

That is, the expectation does not exist.

If a gambler could play repeatedly in a kind-hearted casino, the average of the pay-offs would not converge. That is, there is no law of large numbers in the usual sense that

$$\sum_{i=1}^{n} \frac{X_i}{n} \to \mu \tag{16}$$

However, a generalized law of large numbers does hold for the kind-hearted casino, just as it did for the Petersburg problem (Feller, 1968, pp. 251-253). That is, we have the following

THEOREM. There exists a function of n, $f(n)$, such that

$$\sum_{i=1}^{n} \frac{X_i}{f(n)} \to 1 \qquad \text{i.p.} \tag{17}$$

For the kind-hearted casino, $f(n) = n \ln n$.

PROOF. Use the method of truncation.

For variables $X_1, X_2, \ldots, X_n$ define new variables $Z_1, Z_2, \ldots, Z_n$ such that

$$Z_i = X_i \qquad \text{if} \qquad X \leq n \ln n$$

$$= 0 \qquad \text{otherwise.}$$

Then we have

1. $$P\{Z_1 = X_1, Z_2 = X_2, \ldots, Z_n = X_n\} \to 1 \quad \text{as} \quad n \to \infty,$$

2. $$EZ_i = \sum_{z=1}^{n \ln n} \frac{1}{(z+1)} \sim \ln(n \ln n),$$

3. $$Var Z_i = E(Z_i^2) - (EZ_i)^2 = \sum_{z=1}^{n \ln n} \frac{z^2}{(z+1)(z+2)} - \left[\sum_{z=1}^{n \ln n} \frac{1}{(z+1)}\right]^2$$
$$\sim n \ln n - [\ln(n \ln n)]^2 \sim n \ln n,$$

and thus,

4. $$E\left(\sum_{i=1}^{n} Z_i\right) \sim n\ln(n\ln n) = n[\ln n + \ln^2 n],$$

and

5. $$Var\left(\sum_{i=1}^{n} Z_i\right) \sim n^2 \ln n,$$

and, using Chebyshev's inequality, we have

6. $$\sum_{i=1}^{n} \frac{Z_i}{[n\ln n]} \to 1 \qquad \text{i. p.},$$

and, using 1, we have,

$$\sum_{i=1}^{n} \frac{X_i}{[n\ln n]} \to 1 \qquad \text{i. p.} \qquad Q.E.D.$$

What this says, intuitively, is that $\bar{x}$ increases as $\ln n$. It is easy to calculate the exact distribution for $\sum_{i=1}^{n} X_i$ numerically, since

$$\begin{aligned} P(X = x) &= P(X \geq x) - P(X > x) \\ &= 1/(x+1) - 1/(x+2) \\ &= 1/[(x+1)(x+2)], \end{aligned}$$

and, if $Y_n = \sum_{i=1}^{m} X_i$, we have the convolution

$$P(Y_{n+1} = k) = \sum_{i=0}^{k} P(X = i)P(Y_n = k - i).$$

For $n = 1, 2, 3, \ldots, 10$ we have

n	*Median of Y*
1	0.5
2	2
3	4
4	7
5	10
6	13
7	16
8	19
9	23
10	26

ILLUSTRATION OF THE IDEAS

The ideas developed above are illustrated in two ways. First, biologists sometimes seriously base their theories on the fanciful assumption that animal behavior depends on information that animals cannot have, much like a gambler basing his decisions on a knowledge of the future. Second, the distribution of payoff size for a gambler in a kind-hearted casino very closely resembles the distribution of the number of papers published by individual scientists, and the number of times individual papers are cited in the literature. For such distributions, the generalized law of large numbers suggests that the average number of times per paper that an author's papers are cited should increase with the number of papers cited.

FANCIFUL ASSUMPTIONS BY BIOLOGISTS.

While the idea that a gambler might "quit while he is ahead," or play in a "kind-hearted casino," is clearly fanciful, biologists frequently make similar assumptions in constructing mathematical models of habitat use. For example, Fretwell (1972, following Fretwell and Lucas 1970) developed the idea of an "ideal free distribution," based on the assumption that an animal would choose to inhabit the best available habitat, taking into account the density of conspecifics with which it must share that habitat. Cook and Hubbard (1977) assumed that individual insect parasites searching for hosts that are distributed in patches would choose those patches with the highest density of unparasitized hosts, while Comins and Hassell (1979) assumed that predators or parasites would search in such a way as to maximize their rate of encounter with healthy hosts. None of these authors suggest how the animals know which patch is best, or how their behavior depends on experience they could have, or on information they could obtain about the environment.

These authors proceed to draw ecological conclusions based on their assumptions. Fretwell concludes that the densities of animals will be higher in better-quality patches, but the reproductive success of animals in different habitats will be the same. Cook and Hubbard conclude that the density of unparasitized hosts will be the same in all patches, and Comins and Hassell conclude that animals will become aggregated in patches with the most prey or hosts. It is not clear that such conclusions are false, but since they are based on unrealistic assumptions, the arguments that produce the conclusions are invalid.

NUMBER OF PAPERS PUBLISHED, AND NUMBER OF TIMES PAPERS ARE CITED.

The number of scientists publishing exactly n papers has been observed to be roughly proportional to $1/n^2$ (Lotka's law, see Price 1963). This is a discrete version of the Pareto distribution, as is the distribution we obtain for our gambler in a kind-hearted casino (11). The usual law of large numbers fails in each case, but a generalized law holds, with $f(n) = n \ln n$ in each case.

Price (1963) points out several conclusions based on the fact that the distribution of the number of papers per author has such a long tail. One conclusion is that among n authors in a field, about half of the papers will be written by the $n^{1/2}$ most prolific, each of whom will have written at least $n^{1/2}$ papers. If eminence is then defined by whether a scientist has published at least $n^{1/2}$ papers, one sees that the number of eminent scientists increases only half as fast as the number of scientists. I think this is amusing, but it is only numerology.

The number of times individual papers are cited follows a distribution similar to that of the numbers of papers written by individual authors. It is interesting that of approximately 19 million papers cited in the period from 1961 to 1980, most were cited only once (Garfield 1984). As a consequence of our generalized law of large numbers we would expect that for a number of cited papers (chosen at random with respect to the number of citations), the average number of citations would increase with the number of papers. This would be true for any long-tailed distribution resembling the Pareto, whether the usual law of large numbers held, or only a generalized law of large numbers held, or even if neither held. It has been noted that as the number of papers published by an individual author increases beyond a certain number the proportion of papers that are cited decreases (Menard 1971). This suggests that there is a practical limit to the number of useful ideas an author will have, even if there is practically no limit to the number of papers he can publish.

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