Bayesian networks

Chapter 14.1–3

Outline

◇ Syntax

◇ Semantics

◇ Parameterized distributions

Bayesian networks

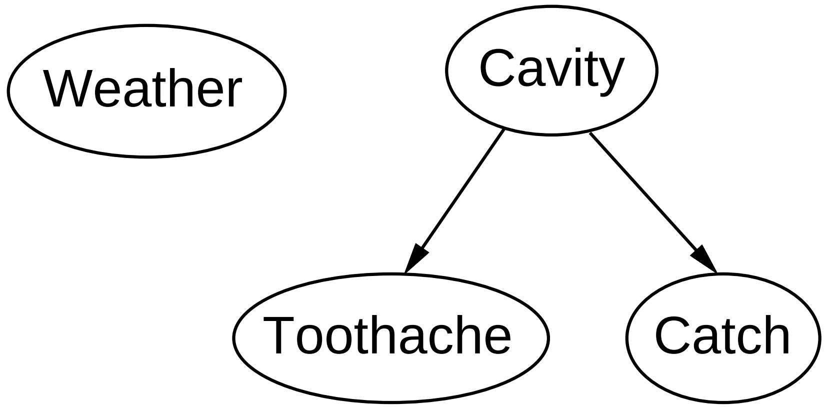
A simple, graphical notation for conditional independence assertions
and hence for compact specification of full joint distributions

Syntax:
- a set of nodes, one per variable
- a directed, acyclic graph (link $\approx$ "directly influences")
- a conditional distribution for each node given its parents:
  $\mathbf{P}(X_i|Parents(X_i))$

In the simplest case, conditional distribution represented as
a conditional probability table (CPT) giving the
distribution over X_i for each combination of parent values

Example

Topology of network encodes conditional independence assertions:



$Weather$ is independent of the other variables

$Toothache$ and $Catch$ are conditionally independent given $Cavity$

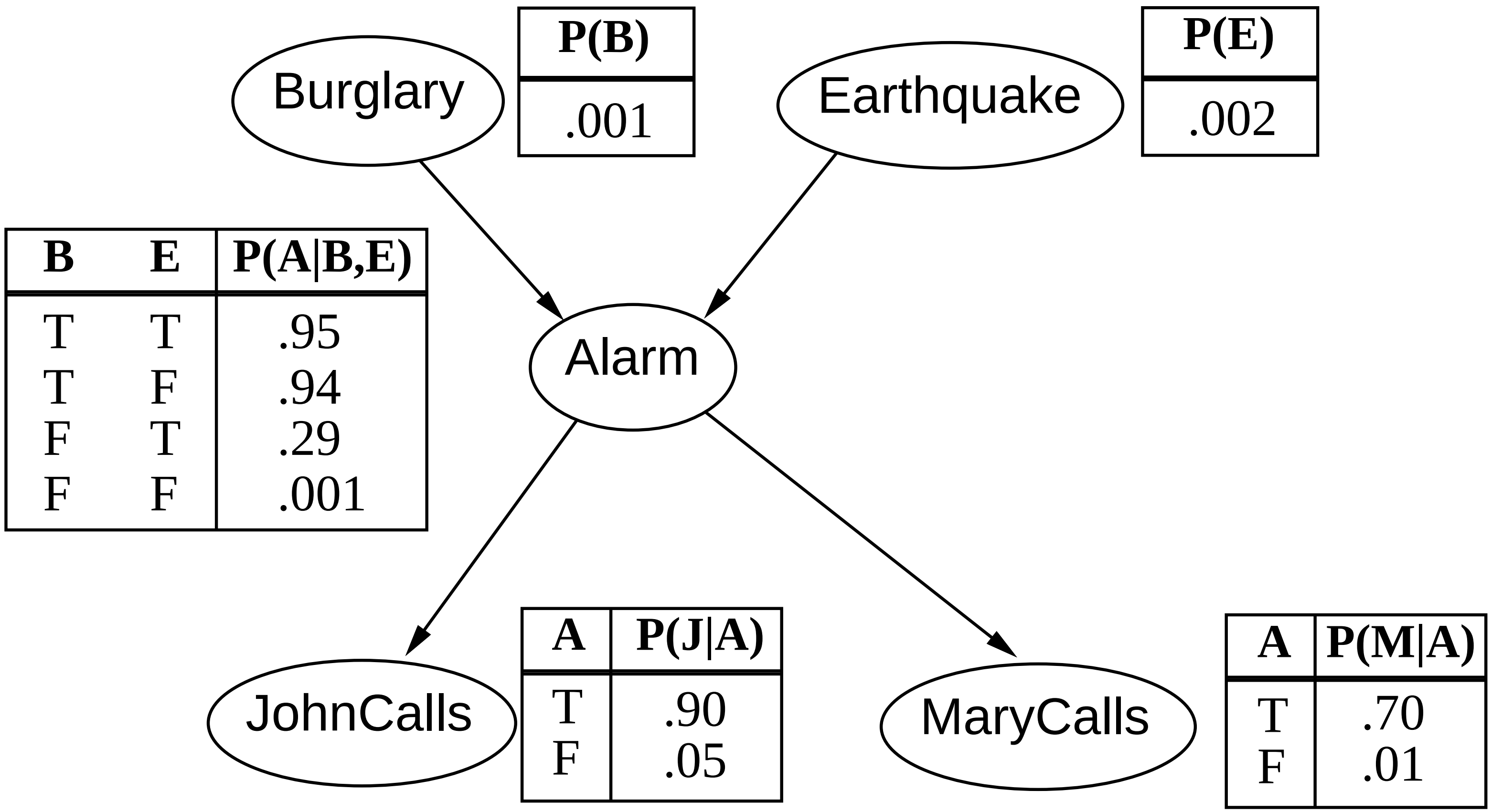
Example

I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

Variables: $Burglar$, $Earthquake$, $Alarm$, $JohnCalls$, $MaryCalls$
Network topology reflects "causal" knowledge:

– A burglar can set the alarm off
– An earthquake can set the alarm off
– The alarm can cause Mary to call
– The alarm can cause John to call

Example contd.



P(B)
.001

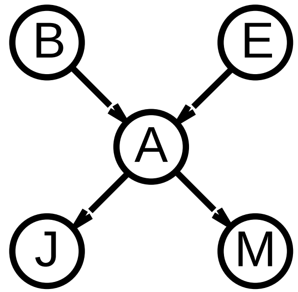
P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

Compactness



A CPT for Boolean X_i with k Boolean parents has
2^k rows for the combinations of parent values

Each row requires one number p for $X_i = true$
(the number for $X_i = false$ is just $1 - p$)

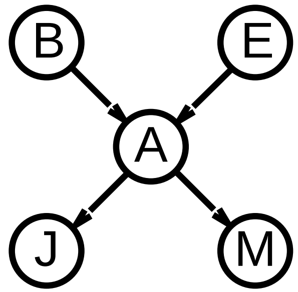
If each variable has no more than k parents,
the complete network requires $O(n \cdot 2^k)$ numbers

I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution

For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)

Global semantics

Global semantics defines the full joint distribution
as the product of the local conditional distributions:



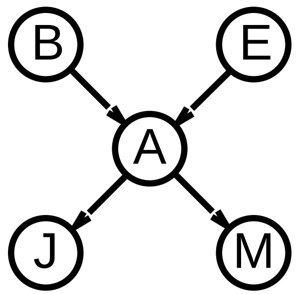
$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

e.g., $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$=$

Global semantics

"Global" semantics defines the full joint distribution as the product of the local conditional distributions:



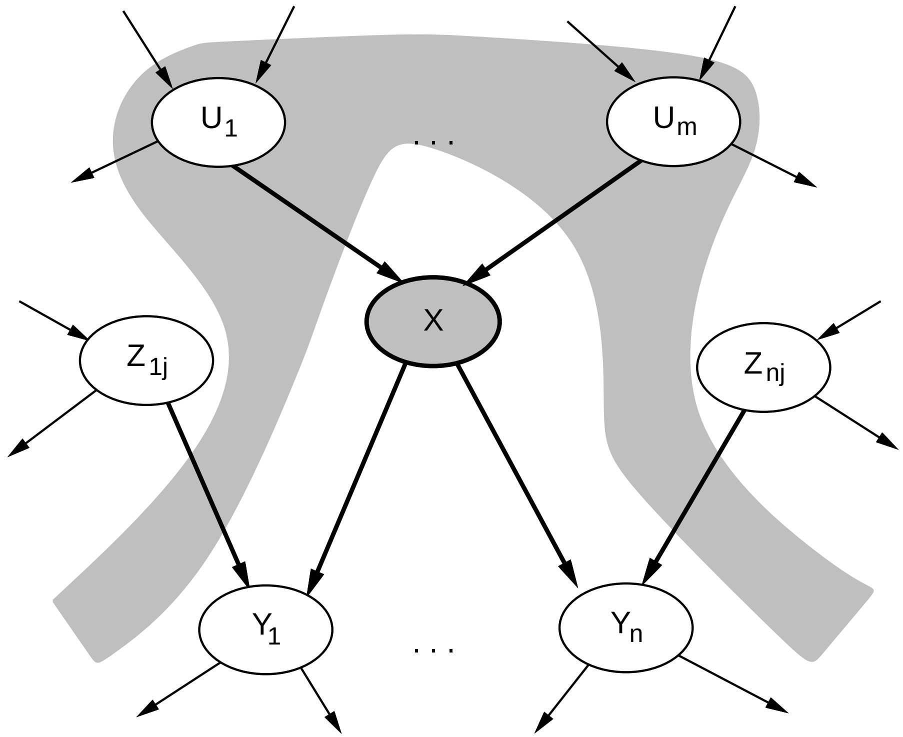
$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

e.g., $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$$\begin{aligned} &= P(j|a)P(m|a)P(a|\neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$

Local semantics

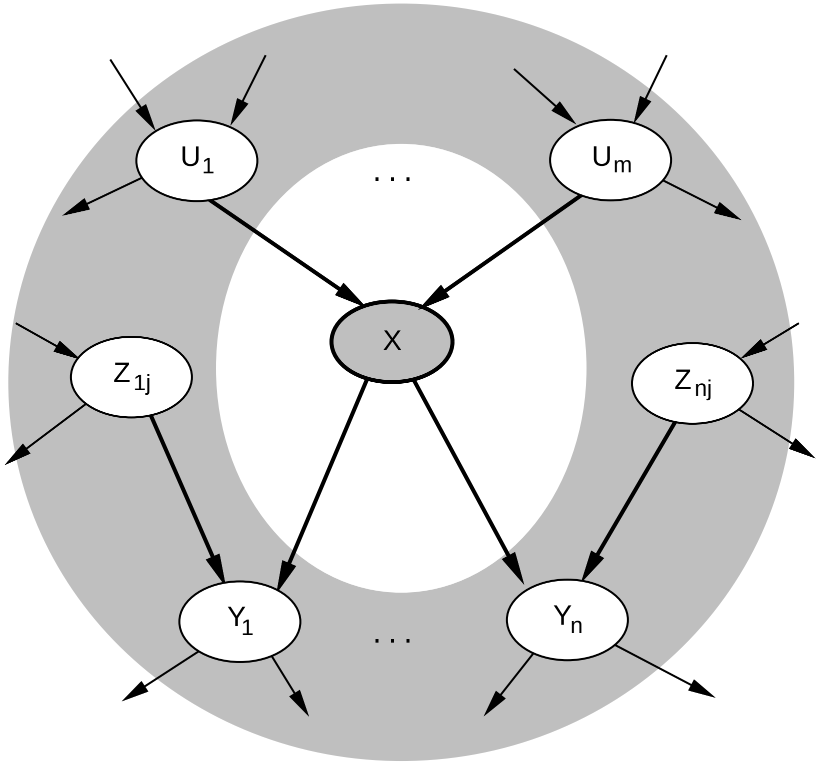
Local semantics: each node is conditionally independent of its nondescendants given its parents



Theorem: Local semantics $\Leftrightarrow$ global semantics

Markov blanket

Each node is conditionally independent of all others given its Markov blanket: parents + children + children's parents



Constructing Bayesian networks

Need a method such that a series of locally testable assertions of conditional independence guarantees the required global semantics

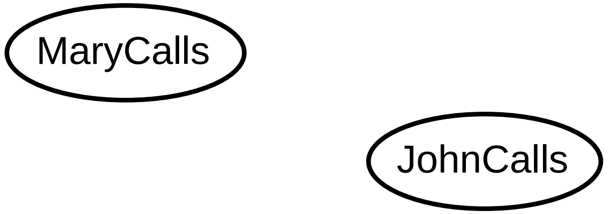
1. Choose an ordering of variables $X_1, \ldots, X_n$
2. For $i = 1$ to n
   add X_i to the network
   select parents from $X_1, \ldots, X_{i-1}$ such that
   $\mathbf{P}(X_i | Parents(X_i)) = \mathbf{P}(X_i | X_1, \ldots, X_{i-1})$

This choice of parents guarantees the global semantics:

$$\begin{aligned} \mathbf{P}(X_1, \ldots, X_n) &= \prod_{i=1}^{n} \mathbf{P}(X_i | X_1, \ldots, X_{i-1}) \quad \text{(chain rule)} \\ &= \prod_{i=1}^{n} \mathbf{P}(X_i | Parents(X_i)) \quad \text{(by construction)} \end{aligned}$$

Example

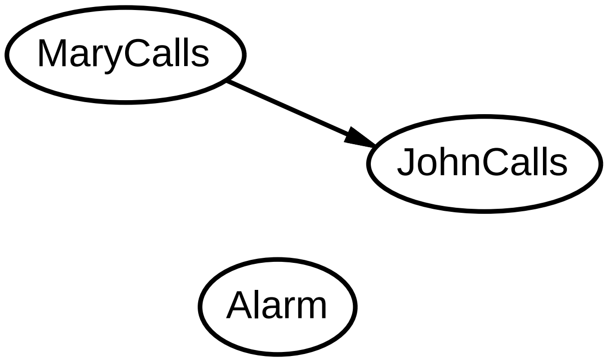
Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$?

Example

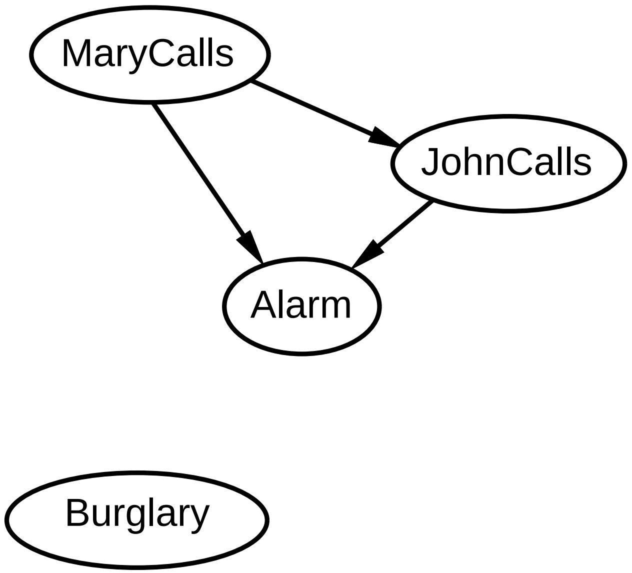
Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No
$P(A|J,M) = P(A|J)$? $P(A|J,M) = P(A)$?

Example

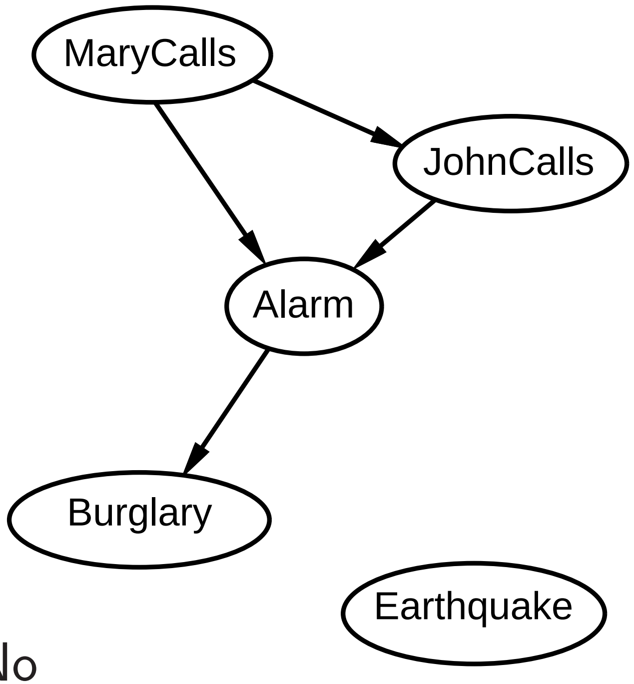
Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No
$P(A|J,M) = P(A|J)$? $P(A|J,M) = P(A)$? No
$P(B|A,J,M) = P(B|A)$?
$P(B|A,J,M) = P(B)$?

Example

Suppose we choose the ordering M, J, A, B, E



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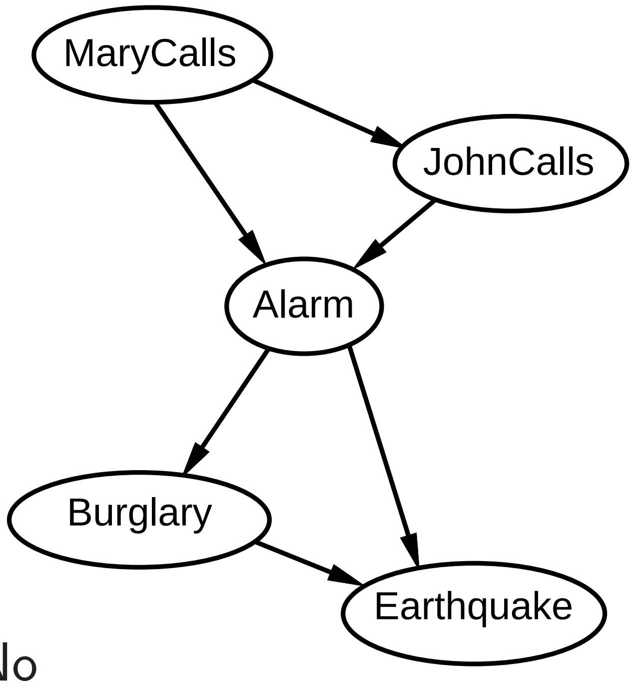
$P(B|A,J,M) = P(B|A)$? Yes
$P(B|A,J,M) = P(B)$? No
$P(E|B,A,J,M) = P(E|A)$?
$P(E|B,A,J,M) = P(E|A,B)$?

Example

Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No
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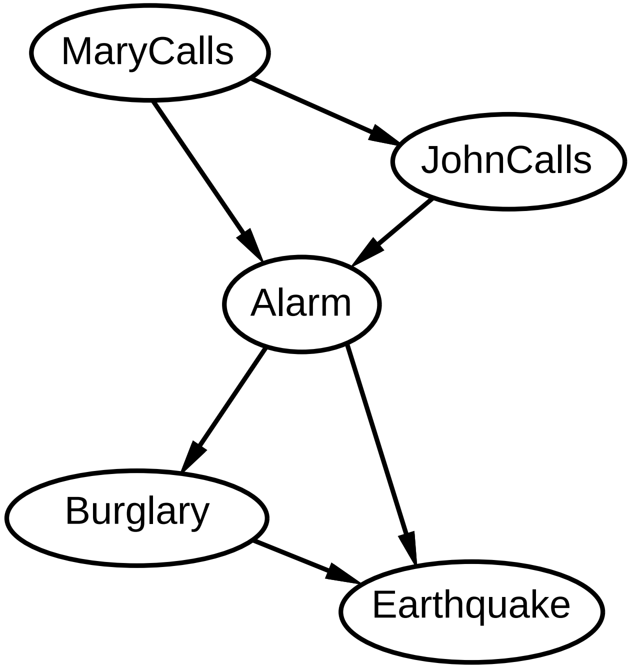
$P(B|A, J, M) = P(B)$? No
$P(E|B, A, J, M) = P(E|A)$? No
$P(E|B, A, J, M) = P(E|A, B)$? Yes

Example contd.



Deciding conditional independence is hard in noncausal directions

(Causal models and conditional independence seem hardwired for humans!)

Assessing conditional probabilities is hard in noncausal directions

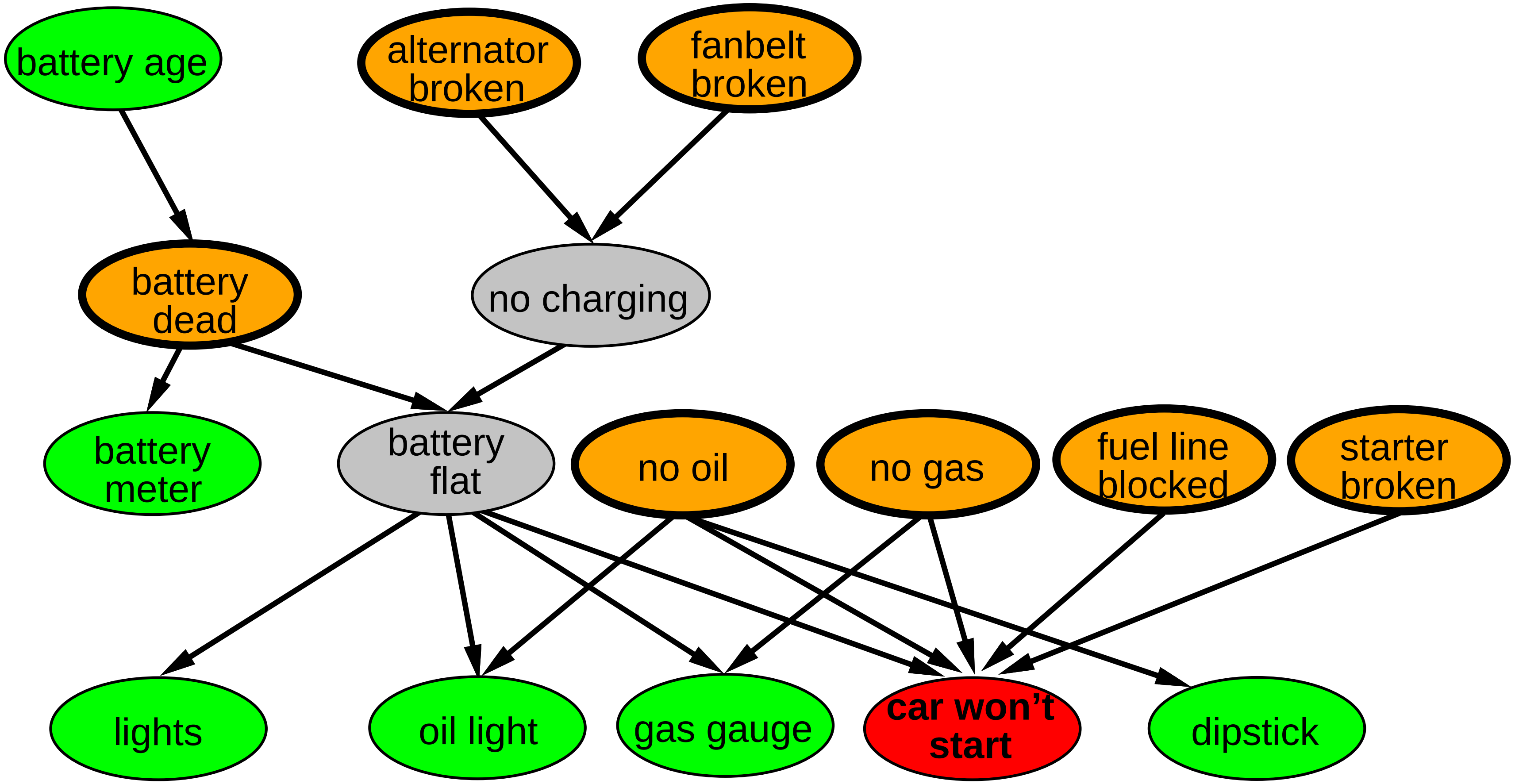
Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed

Example: Car diagnosis

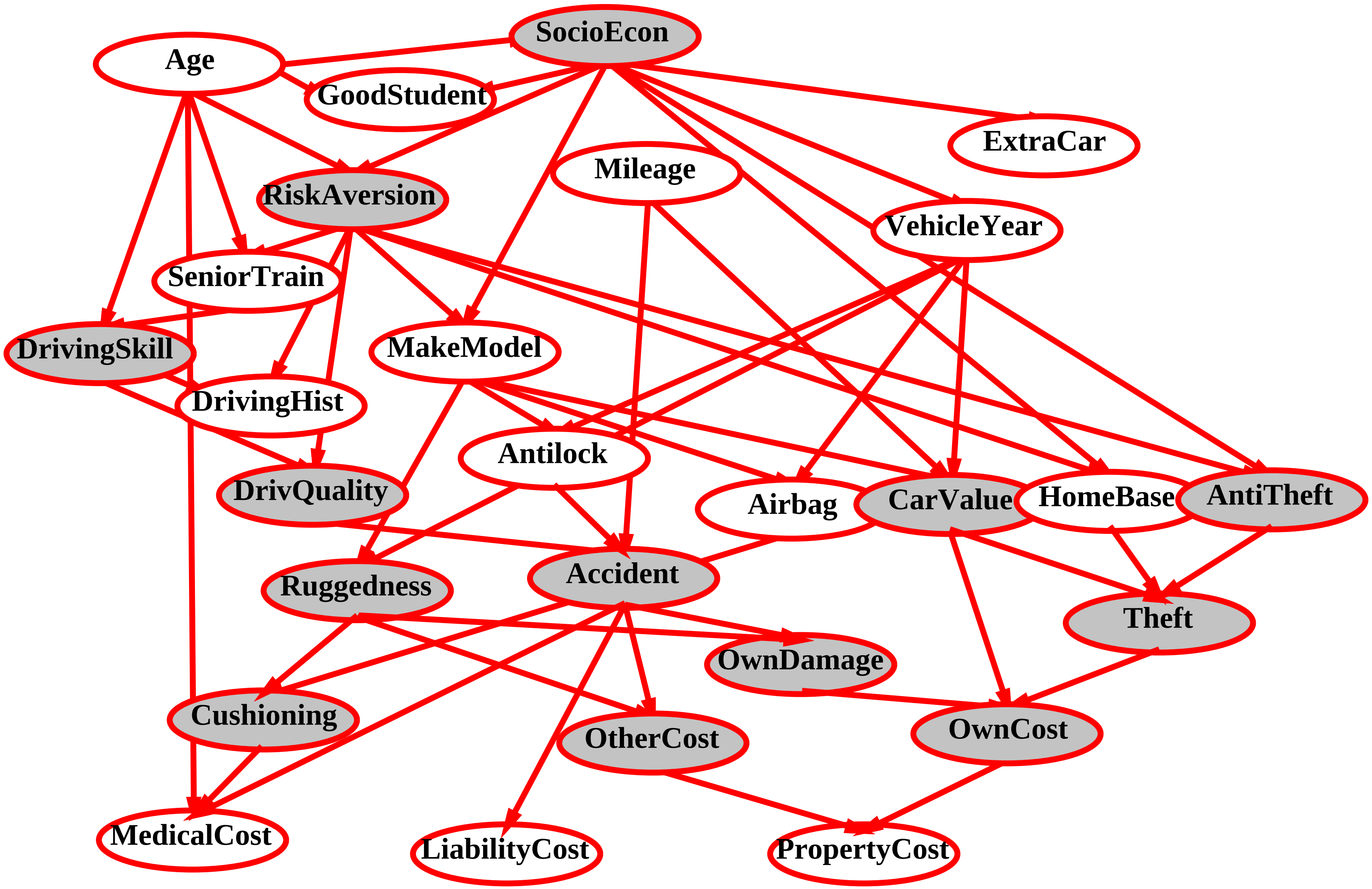
Initial evidence: car won't start

Testable variables (green), "broken, so fix it" variables (orange)

Hidden variables (gray) ensure sparse structure, reduce parameters



Example: Car insurance



Compact conditional distributions

CPT grows exponentially with number of parents
CPT becomes infinite with continuous-valued parent or child

Solution: canonical distributions that are defined compactly

Deterministic nodes are the simplest case:
$X = f(Parents(X))$ for some function f

E.g., Boolean functions
$NorthAmerican \Leftrightarrow Canadian \vee US \vee Mexican$

E.g., numerical relationships among continuous variables

$$\frac{\partial Level}{\partial t} = \text{ inflow + precipitation - outflow - evaporation}$$

Compact conditional distributions contd.

Noisy-OR distributions model multiple noninteracting causes

1) Parents $U_1 \ldots U_k$ include all causes (can add leak node)

2) Independent failure probability q_i for each cause alone

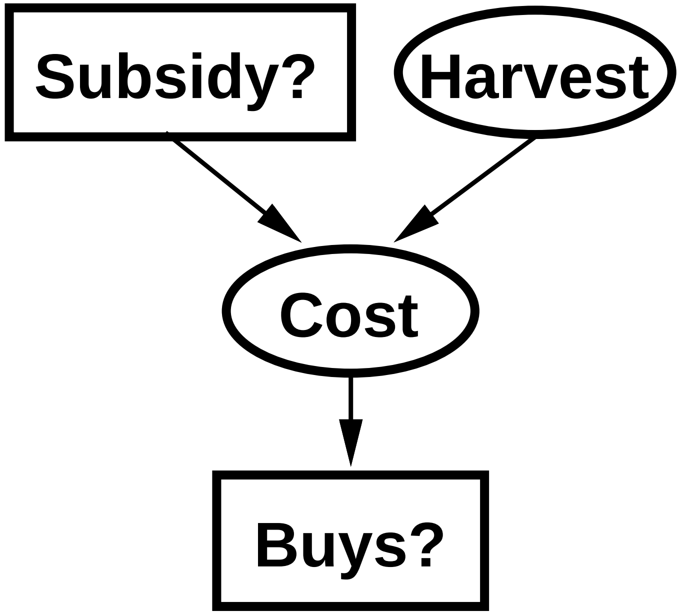
$$\Rightarrow\ P(X|U_1 \ldots U_j, \neg U_{j+1} \ldots \neg U_k) = 1 - \prod_{i=1}^{j} q_i$$

$Cold$	Flu	$Malaria$	$P(Fever)$	$P(\neg Fever)$
F	F	F	$\mathbf{0.0}$	1.0
F	F	T	0.9	$\mathbf{0.1}$
F	T	F	0.8	$\mathbf{0.2}$
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	$\mathbf{0.6}$
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

Number of parameters **linear** in number of parents

Hybrid (discrete+continuous) networks

Discrete (*Subsidy?* and *Buys?*); continuous (*Harvest* and *Cost*)



Option 1: discretization—possibly large errors, large CPTs
Option 2: finitely parameterized canonical families

1) Continuous variable, discrete+continuous parents (e.g., *Cost*)
2) Discrete variable, continuous parents (e.g., *Buys?*)

Continuous child variables

Need one conditional density function for child variable given continuous parents, for each possible assignment to discrete parents

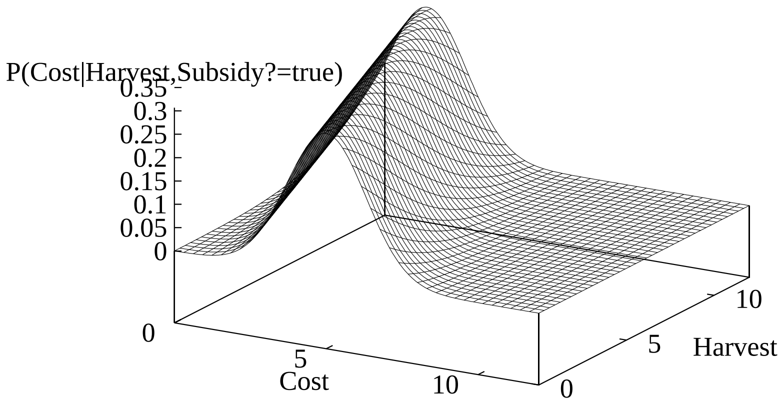
Most common is the linear Gaussian model, e.g.,:

$$
\begin{aligned}
& P(Cost = c | Harvest = h, Subsidy? = true) \\
&= N(a_t h + b_t, \sigma_t)(c) \\
&= \frac{1}{\sigma_t \sqrt{2\pi}} exp\left(-\frac{1}{2} \left(\frac{c - (a_t h + b_t)}{\sigma_t} \right)^2 \right)
\end{aligned}
$$

Mean $Cost$ varies linearly with $Harvest$, variance is fixed

Linear variation is unreasonable over the full range
but works OK if the **likely** range of $Harvest$ is narrow

Continuous child variables



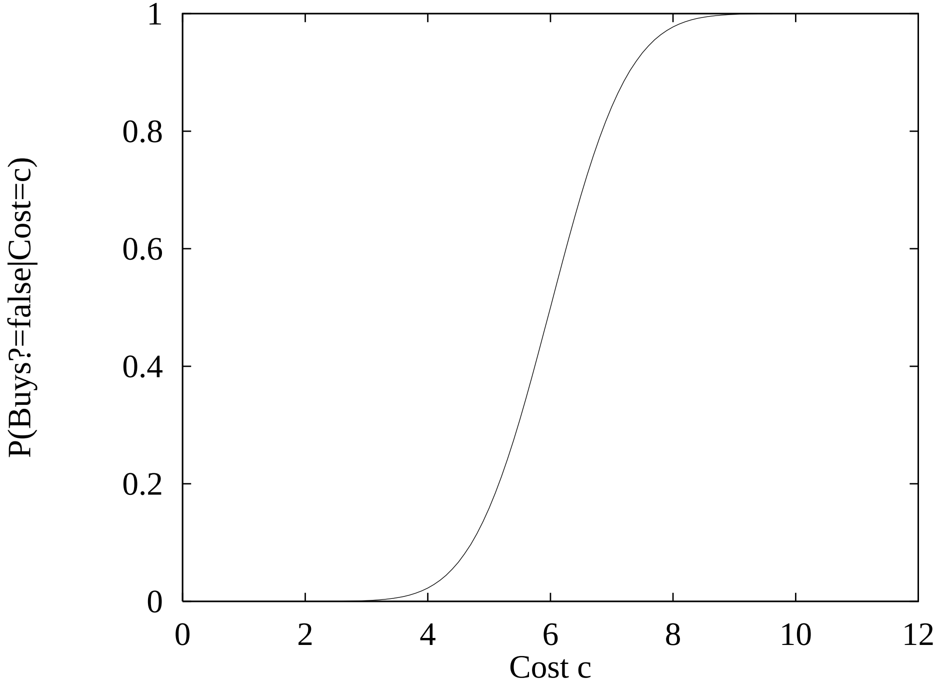
All-continuous network with LG distributions

$\Rightarrow$ full joint distribution is a multivariate Gaussian

Discrete+continuous LG network is a conditional Gaussian network i.e., a multivariate Gaussian over all continuous variables for each combination of discrete variable values

Discrete variable w/ continuous parents

Probability of $Buys?$ given $Cost$ should be a "soft" threshold:



Probit distribution uses integral of Gaussian:

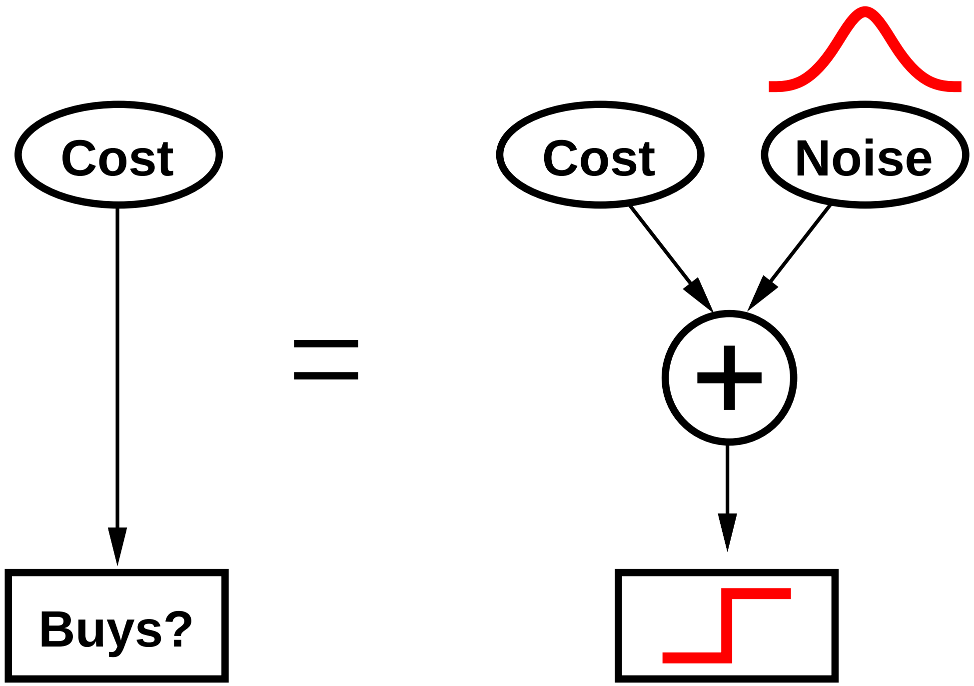
$$\Phi(x) = \int_{-\infty}^{x} N(0,1)(x)dx$$

$$P(Buys?=true \mid Cost=c) = \Phi((-c+\mu)/\sigma)$$

Why the probit?

1. It's sort of the right shape

2. Can view as hard threshold whose location is subject to noise

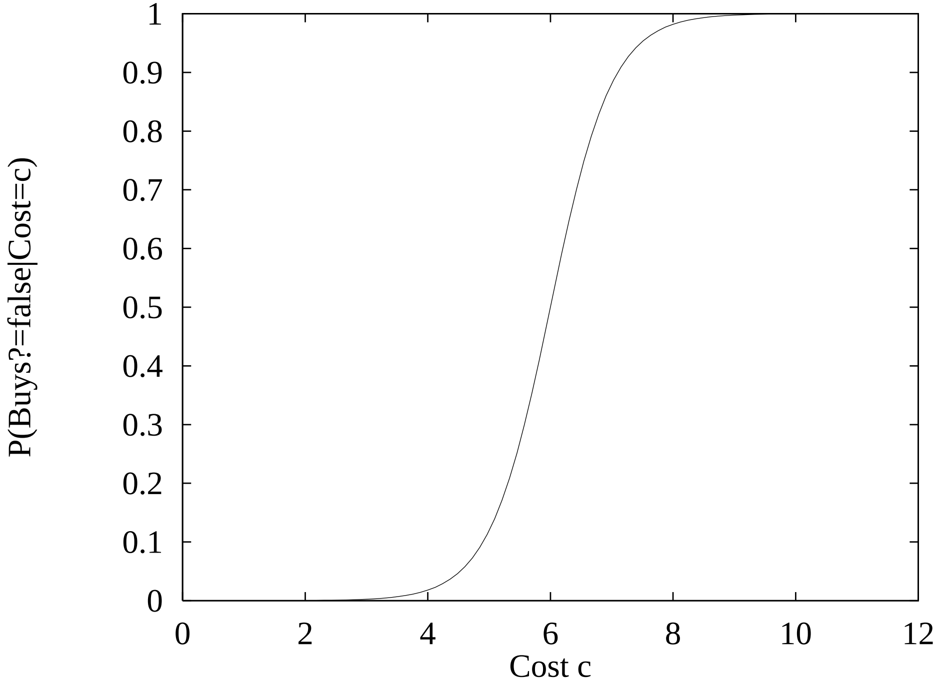


Discrete variable contd.

Sigmoid (or logit) distribution also used in neural networks:

$$P(Buys? = true \mid Cost = c) = \frac{1}{1 + exp(-2\frac{-c+\mu}{\sigma})}$$

Sigmoid has similar shape to probit but much longer tails:



Summary

Bayes nets provide a natural representation for (causally induced) conditional independence

Topology + CPTs = compact representation of joint distribution

Generally easy for (non)experts to construct

Canonical distributions (e.g., noisy-OR) = compact representation of CPTs

Continuous variables $\Rightarrow$ parameterized distributions (e.g., linear Gaussian)