Reinforcement Learning

- Control learning
- Control polices that choose optimal actions
- Q learning
- Convergence

Control Learning

Consider learning to choose actions, e.g.,

- Robot learning to dock on battery charger
- Learning to choose actions to optimize factory output
- Learning to play Backgammon

Note several problem characteristics

- Delayed reward
- Opportunity for active exploration
- Possibility that state only partially observable
- Possible need to learn multiple tasks with same sensors/effectors

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One Example: TD-Gammon

Tesauro, 1995

Learn to play Backgammon

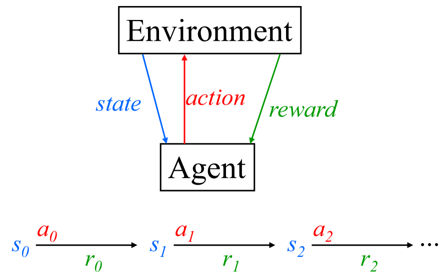
Immediate reward

- +100 if win
- -100 if lose
- 0 for all other states

Trained by playing 1.5 million games against itself

Now approximately equal to best human player

Reinforcement Learning Problem



Goal: learn to choose actions that maximize

$r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots$, where $0 \le \gamma < 1$

Markov Decision Process

Assume

- finite set of states S
- set of actions A
- at each discrete time, agent observes state $s_t \in S$ and choose action $a_t \in A$
- then receives immediate reward r_t
- and state changes to s_{t+1}
- Markov assumption: $s_{t+1} = \delta(s_t, a_t)$ and $r_t = r(s_t, a_t)$
  - i.e., r_t and s_{t+1} depend only on current state and action
  - functions δ and r may be nondeterministic
  - functions δ and r no necessarily known to agent

Agent's Learning Task

Execute action in environment, observe results, and

- learn action policy $\pi : S \rightarrow A$ that maximizes

$$E[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \ldots]$$

from any starting state in S

- here $0 \le \gamma < 1$ is the *discount factor* for future rewards

Note something new:

- target function is $\pi : S \rightarrow A$
- but we have no training examples of form <*s*,*a*>
- training examples are of form <<*s*,*a*>,*r*>

Value Function

To begin, consider deterministic worlds …

For each possible policy π the agent might adopt, we can define an evaluation function over states

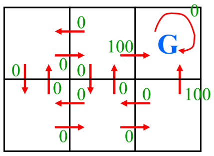
$$V^{\pi}(s) \equiv r_t + \gamma\, r_{t+1} + \gamma^{2} r_{t+2} + \ldots$$

$$\equiv \sum_{i=0}^{\infty} \gamma^{i} r_{t+i}$$

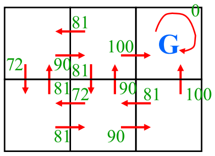
where $r_t, r_{t+1}, \ldots$ are generated by following policy π starting at state s

Restated, the task is to learn the optimal policy π^*

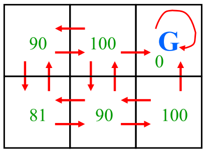
$$\pi^* \equiv \underset{\pi}{\operatorname{argmax}}\, V^{\pi}(s), (\forall s)$$



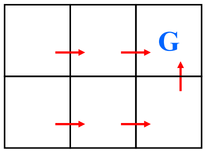
r(s,a) (immediate reward) values



Q(s,a) values



V(s)* values



One optimal policy

What to Learn

We might try to have agent learn the evaluation function V^{π^*} (which we write as V^*)

We could then do a lookahead search to choose best action from any state s because

$$\pi^*(s) \equiv \underset{a}{\operatorname{argmax}}\left[r(s,a) + \gamma\, V^*(\delta(s,a))\right]$$

A problem:

- This works well if agent knows a $\delta : S \times A \rightarrow S$, and $r : S \times A \rightarrow \Re$
- But when it doesn't, we can't choose actions this way

Q Function

Define new function very similar to V*

$$Q(s,a) \equiv r(s,a) + \gamma\, V^*(\delta(s,a))$$

If agent learns Q, it can choose optimal action even without knowing d!

$$\pi^*(s) \equiv \underset{a}{\operatorname{argmax}}\left[r(s,a) + \gamma\, V^*(\delta(s,a))\right]$$

$$\pi^*(s) \equiv \underset{a}{\operatorname{argmax}}\, Q(s,a)$$

Q is the evaluation function the agent will learn

Training Rule to Learn *Q*

Note *Q* and *V** closely related:

$$V^*(s) = \max_{a'} Q(s,a')$$

Which allows us to write *Q* recursively as

$$Q(s_t,a_t) = r(s_t,a_t) + \gamma\, V^*(\delta(s_t,a_t))$$

$$= r(s_t,a_t) + \gamma\, \max_{a'} Q(s_{t+1},a')$$

Let $\hat{Q}$ denote learner's current approximation to *Q*. Consider training rule

$$\hat{Q}(s,a) \leftarrow r + \gamma\, \max_{a'} \hat{Q}(s',a')$$

where *s'* is the state resulting from applying action *a* in state *s*

Q Learning for Deterministic Worlds

For each *s,a* initialize table entry $\hat{Q}(s,a) \leftarrow 0$

Observe current state *s*

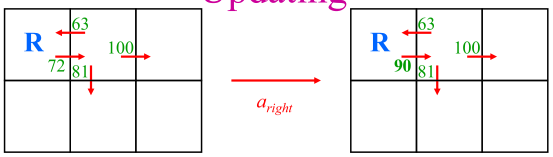
Do forever:

- Select an action *a* and execute it
- Receive immediate reward *r*
- Observe the new state *s'*
- Update the table entry for $\hat{Q}(s,a)$ as follows:

$$\hat{Q}(s,a) \leftarrow r + \gamma\, \max_{a'} \hat{Q}(s',a')$$

- $s \leftarrow s'$

Updating



initial state: s_1

next state: s_2

$$\hat{Q}(s_1, a_{right}) \leftarrow r + \gamma \max_{a'} \hat{Q}(s_2, a')$$

$$\leftarrow 0 + 0.9 \max\{63, 81, 100\} = 90$$

notice if rewards non-negative, then

$$(\forall s, a, n)\ \hat{Q}_{n+1}(s,a) \geq \hat{Q}_n(s,a)$$

and

$$(\forall s, a, n)\ 0 \leq \hat{Q}_n(s,a) \leq Q(s,a)$$

Convergence

$\hat{Q}$ converges to Q. Consider case of deterministic world where each <*s*,*a*> visited infinitely often.

Proof: define a full interval to be an interval during which each <*s*,*a*> is visited. During each full interval the largest error in $\hat{Q}$ table is reduced by factor of γ

Let $\hat{Q}_n$ be table after *n* updates, and Δ_n be the maximum error in $\hat{Q}_n$; that is

$$\Delta_n = \max_{s,a} \left| \hat{Q}_n(s,a) - Q(s,a) \right|$$

Convergence (cont)

For any table entry $\hat{Q}_n(s,a)$ updated on iteration *n+1*, the error in the revised estimate $\hat{Q}_n(s,a)$ is

$$\left|\hat{Q}_{n+1}(s,a) - Q(s,a)\right| = \left|(r + \gamma \max_{a'} \hat{Q}_n(s',a')) - (r + \gamma \max_{a'} Q(s',a'))\right|$$

$$= \gamma \left| \max_{a'} \hat{Q}_n(s',a') - \max_{a'} Q(s',a') \right|$$

$$\leq \gamma \max_{a'} \left| \hat{Q}_n(s',a') - Q(s',a') \right|$$

$$\leq \gamma \max_{s'',a'} \left| \hat{Q}_n(s'',a') - Q(s'',a') \right|$$

$$\left|\hat{Q}_{n+1}(s,a) - Q(s,a)\right| \leq \gamma\ \Delta_n$$

Note we used general fact that

$$\left| \max_a f_1(a) - \max_a f_2(a) \right| \leq \max_a \left| f_1(a) - f_2(a) \right|$$

Nondeterministic Case

What if reward and next state are non-deterministic?

We redefine *V*,*Q* by taking expected values

$$V^{\pi}(s) \equiv \mathrm{E}[r_t + \gamma\ r_{t+1} + \gamma^2 r_{t+2} + \ldots]$$

$$\equiv \mathrm{E}\left[\sum_{i=0}^{\infty} \gamma^i r_{t+i}\right]$$

$$Q(s,a) \equiv \mathrm{E}[r(s,a) + \gamma\ V^*(\delta\ (s,a))]$$

Nondeterministic Case

Q learning generalizes to nondeterministic worlds

Alter training rule to

$$\hat{Q}_n(s,a) \leftarrow (1 - \alpha_n)\hat{Q}_{n-1}(s,a) + \alpha_n[r + \max_{a'} \hat{Q}_{n-1}(s',a')]$$

where

$$\alpha_n = \frac{1}{1 + visits_n(s,a)}$$

Can still prove converge of $\hat{Q}$ to Q [Watkins and Dayan, 1992]

Temporal Difference Learning

Q learning: reduce discrepancy between successive *Q* estimates

One step time difference:

$$Q^{(1)}(s_t, a_t) \equiv r_t + \gamma \max_a \hat{Q}(s_{t+1}, a)$$

Why not two steps?

$$Q^{(2)}(s_t, a_t) \equiv r_t + \gamma\ r_{t+1} + \gamma^2 \max_a \hat{Q}(s_{t+2}, a)$$

Or *n*?

$$Q^{(n)}(s_t, a_t) \equiv r_t + \gamma\ r_{t+1} + \ldots + \gamma^{n-1} r_{t+n-1} + \gamma^n \max_a \hat{Q}(s_{t+n}, a)$$

Blend all of these:

$$Q^{\lambda}(s_t, a_t) \equiv (1 - \lambda)\left[Q^{(1)}(s_t, a_t) + \lambda\ Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t) + \ldots\right]$$

Temporal Difference Learning

$$Q^{\lambda}(s_t,a_t) \equiv (1-\lambda)\left[Q^{(1)}(s_t,a_t)+\lambda\, Q^{(2)}(s_t,a_t)+\lambda^{2} Q^{(3)}(s_t,a_t)+\ldots\right]$$

Equivalent expression:

$$Q^{\lambda}(s_t,a_t) \equiv r_t+\gamma\left[(1-\lambda)\max_{a}\hat{Q}(s_t,a_t)+\lambda\, Q^{\lambda}(s_{t+1},a_{t+1})\right]$$

TD(λ) algorithm uses above training rule

- Sometimes converges faster than *Q* learning
- converges for learning V^* for any $0 \le \lambda \le 1$ (Dayan, 1992)
- Tesauro’s TD-Gammon uses this algorithm

Subtleties and Ongoing Research

- Replace $\hat{Q}$ table with neural network or other generalizer
- Handle case where state only partially observable
- Design optimal exploration strategies
- Extend to continuous action, state
- Learn and use d : $S \times A \rightarrow S$, d approximation to δ
- Relationship to dynamic programming

RL Summary

- Reinforcement learning (RL)
  - control learning
  - delayed reward
  - possible that the state is only partially observable
  - possible that the relationship between states/actions unknown
- Temporal Difference Learning
  - learn discrepancies between successive estimates
  - used in TD-Gammon
- V(s) - state value function
  - needs known reward/state transition functions

RL Summary

- Q(s,a) - state/action value function
  - related to V
  - does not need reward/state trans functions
  - training rule
  - related to dynamic programming
  - measure actual reward received for action and future value using current Q function
  - deterministic - replace existing estimate
  - nondeterministic - move table estimate towards measure estimate
  - convergence - can be shown in both cases