Factorial Studies

- Factor A → I levels
- Factor B → J levels
  - I*J Treatment groups
  - $n_{ij}$ values for each treatment combination

<table>
<thead>
<tr>
<th></th>
<th>B=1</th>
<th>B=2</th>
<th>...</th>
<th>B=J</th>
</tr>
</thead>
<tbody>
<tr>
<td>A=1</td>
<td>$n_{11}$</td>
<td>$n_{12}$</td>
<td>$n_{1J}$</td>
<td></td>
</tr>
<tr>
<td>A=2</td>
<td>$n_{21}$</td>
<td>$n_{22}$</td>
<td>$n_{1J}$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A=I</td>
<td>$n_{i1}$</td>
<td>$n_{i2}$</td>
<td>$n_{iJ}$</td>
<td></td>
</tr>
</tbody>
</table>

Equal replication, n replicates, in each treatment group.
- Equal $n_{ij}$'s → Balanced design.
- ! The methods here can be used for balanced data.
- Some of the steps won't necessarily work for unbalanced data.
  - Example 1 in section 8.1 for example is unbalanced.

Factor A = Joint
Factor B = Wood
Y = Joint strength

<table>
<thead>
<tr>
<th></th>
<th>B=Pine</th>
<th>B=Oak</th>
<th>B=Walnut</th>
</tr>
</thead>
<tbody>
<tr>
<td>A=Butt</td>
<td>829, 596</td>
<td><strong>1169</strong></td>
<td>1263, 1029</td>
</tr>
<tr>
<td>A=Beveled</td>
<td>1348, 1207</td>
<td>1518, 1927</td>
<td>2571, 2443</td>
</tr>
<tr>
<td>A=Lap</td>
<td>1000, 859</td>
<td>1295, 1561</td>
<td><strong>1489</strong></td>
</tr>
</tbody>
</table>
Section 4.3  Problem 8

- **Y** = Distance of tennis ball
- **A** = Charge size with I = 2  levels
  - 2.5 ml and 5.0 ml
- **B** = Propellant with J = 3 levels
  - lighter fluid, gasoline and carburetor fluid
- n=5 values for each treatment combination

<table>
<thead>
<tr>
<th>Charge</th>
<th>Lighter Fluid</th>
<th>Gasoline Fluid</th>
<th>Carburetor Fluid</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>58 50 59</td>
<td>76 79 71</td>
<td>90 86 86</td>
</tr>
<tr>
<td></td>
<td>43 49</td>
<td>84 73</td>
<td>79 82</td>
</tr>
<tr>
<td>5.0</td>
<td>65 59 67</td>
<td>96 101 87</td>
<td>107 102 97</td>
</tr>
<tr>
<td></td>
<td>61 68</td>
<td>94 91</td>
<td>91 95</td>
</tr>
</tbody>
</table>

First plot the data!!

It helps to offset the point a bit so they don't end up on top of each other.
Another example:

- \( Y = \) Distance of rubber band car
- \( A = \) Rotations with \( I = 3 \) levels
  - 9, 10, and 10 rotations
- \( B = \) Rubber band size with \( J = 3 \) levels
  - Small, medium, and large
- \( n=4 \) values for each treatment combination

**Interactions and Main Effects**

For the rubber band car example

- The effects of increasing the number of turns are similar
  - No matter what rubber band size is used.
- The responses are fairly parallel.
- In this case there is no important **interaction** between factors Turns and Size.
In general there is no interaction between factors A and B if:

- Effects of factor A are the same for each level of B.
  - Or equivalently
- Effects of factor B are the same for each level of A.
  - Or equivalently
- We have parallel profiles.

- Interactions are like drug interactions.
  - Two drugs have an interaction if the effect of one drug is not the same with or without the other drug.
  - Potentially the effect of a sleeping pill can be different depending on whether you have been drinking alcohol.
  - "Never drink alcohol near the time when you take a sleeping pill. Never drink alcohol in an attempt to fall asleep faster. Not only will alcohol disrupt your sleep even more, it can dangerously interact with the sleeping pill."

http://helpguide.org/life/sleep_aids_medication_insomnia_treatment.htm
With no interactions, it's sensible to summarize the effect of rubber band size

- Using **main effects** for Size
- The averages for each size averaged over all levels of turn.

<table>
<thead>
<tr>
<th>Size</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Large</td>
</tr>
<tr>
<td>2</td>
<td>Medium</td>
</tr>
<tr>
<td>3</td>
<td>Small</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Difference</th>
<th>95% Confidence Limits for Mean(i) - Mean(j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 vs 2 Large - Medium</td>
<td>881</td>
</tr>
<tr>
<td>1 vs 3 Large - Small</td>
<td>2158</td>
</tr>
<tr>
<td>2 vs 3 Medium - Small</td>
<td>1277</td>
</tr>
</tbody>
</table>

Later we will show how to find these confidence intervals.
Similarly, with no interaction, it's sensible to summarize the effect of rubber band turns:

- Using **main effects** for Turns
- The averages for each size averaged over all levels of size.

<table>
<thead>
<tr>
<th>Turns</th>
<th>MEAN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
</tbody>
</table>

Difference Between Means

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>Difference</th>
<th>95% Confidence Limits for Mean(i) - Mean(j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>-290.00</td>
<td>-652 73</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>-629.75</td>
<td>-993 -267</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-339.75</td>
<td>-703 23</td>
</tr>
</tbody>
</table>
An Example with Interactions

- The effect of temperature is not the same for all brands
  - Spider line has a smaller increase in breaking strength with increased temperature.
- The brand effect depends on temperature.
  - There is more difference between brands at the higher temperature.
- The responses are not parallel.
- In this case there is an important interaction between factors Brand and Temperature.
- In this case it would not make sense to summarize the difference between Spider line and Fire line averaging over both temperatures, that is using Line main effects.
  - We would need to give separate confidence intervals for the differences between lines for temp=34° and temp=70°.
  - Having interactions complicates summarizing the results.
Example: Chapter 4 Exercises, Problem 20. A $2^3$ design

- $Y =$ Distance of paper airplane flight
- $A =$ Plane design: Straight wing and T wing
- $B =$ Nose Weight: None and Paper clip
- $C =$ Paper: Notebook and Construction
- $n=2$ values for each treatment combination

There appears to be an $A*C$ interaction.
- The difference between straight wing and T-wing is greater for notebook paper than for construction paper.
  - Or equivalently, the difference between notebook paper and construction paper is greater for T wing than for Straight wing.
  - The responses are not particularly parallel.

- $A*B*C$ interaction:
  - The interaction is somewhat similar for $B=none$ and $B=clip$, the squares and diamonds.
  - If the $A*C$ interaction is the same for all levels of $B$, then there is no $A*B*C$ interaction, a 3-way interaction.
  - There is no $A*B*C$ interaction if
    - The $A*B$ interaction is the same for all levels of $C$.
    - The $A*C$ interaction is the same for all levels of $B$.
    - The $B*C$ interaction is the same for all levels of $A$.

- The two values for the notebook paper T-wing planes with no clip are far apart.
  - More likely the smaller flight time is the odd point.
  - Check the data input as in textile example.
  - Maybe design a phase 2 experiment to look more carefully.
If we did have an A*B*C interaction this would make summarizing the results even more complicated.

- It's nicest to have no interactions.
  - The results are easier to summarize.
  - We have found results that apply more generally over different conditions.

### Software

- Following section 9.3.2 in the book, we can run the factorial analysis with Excel regression.
  - For 2 factor perfectly balanced data, you can use the Excel 2 factor data analysis.
    - This doesn't give residuals or other useful output.
- Other more sophisticated software is really helpful for more complicated regression or ANOVA analyses.
  - Of the graphical user interface program, my favorite is SAS JMP.
  - Minitab is also nice.
  - Most often I use the full blown SAS package.
    - Steeper learning curve.
  - R software is becoming popular as well.
    - R is the open source, free version of S.
    - S was Bell Labs' statistical package.
      - They liked short names like C=Complier and S=Statistics
    - R is free, really good, flexible software but usually requires writing simple code.

For the $2^3$ paper airplane data, using the regression tool in Excel as in section 9.3.2:

**ANOVA**

<table>
<thead>
<tr>
<th></th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>Significance F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>7</td>
<td>251.78</td>
<td>35.97</td>
<td>14.23</td>
<td>0.00061</td>
</tr>
<tr>
<td>Residual</td>
<td>8</td>
<td>20.22</td>
<td>2.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>15</td>
<td>272.00</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Coefficients**

<table>
<thead>
<tr>
<th></th>
<th>Coefficients</th>
<th>Standard Error</th>
<th>t Stat</th>
<th>P-value</th>
<th>Lower 95%</th>
<th>Upper 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>8.109</td>
<td>0.397</td>
<td>20.404</td>
<td>0.000</td>
<td>7.193</td>
<td>9.026</td>
</tr>
<tr>
<td>A</td>
<td>2.047</td>
<td>0.397</td>
<td>5.150</td>
<td>0.00087</td>
<td>1.130</td>
<td>2.963</td>
</tr>
<tr>
<td>B</td>
<td>0.672</td>
<td>0.397</td>
<td>1.691</td>
<td>0.129</td>
<td>-0.245</td>
<td>1.588</td>
</tr>
<tr>
<td>C</td>
<td>-2.922</td>
<td>0.397</td>
<td>-7.352</td>
<td>0.000080</td>
<td>-3.838</td>
<td>-2.005</td>
</tr>
<tr>
<td>A*B</td>
<td>0.234</td>
<td>0.397</td>
<td>0.590</td>
<td>0.572</td>
<td>-0.682</td>
<td>1.151</td>
</tr>
<tr>
<td>A*C</td>
<td>-1.422</td>
<td>0.397</td>
<td>-3.578</td>
<td>0.0072</td>
<td>-2.338</td>
<td>-0.505</td>
</tr>
<tr>
<td>B*C</td>
<td>-0.672</td>
<td>0.397</td>
<td>-1.691</td>
<td>0.129</td>
<td>-1.588</td>
<td>0.245</td>
</tr>
<tr>
<td>A<em>B</em>C</td>
<td>-0.172</td>
<td>0.397</td>
<td>-0.432</td>
<td>0.677</td>
<td>-1.088</td>
<td>0.745</td>
</tr>
</tbody>
</table>
• For ANOVA the effects of factors are described as deviations from the overall mean.
  o Overall mean :  8.109
• For A effects
  o Mean for A=1: 10.156
    ▪ Effect for A=1: 10.156 – 8.109 = 2.047
    ▪ On average straight winged planes flew 2.047 feet farther than average.
  o Mean for A=2:  6.063
    ▪ On average straight winged planes flew 2.047 feet shorter than average.
    ▪ Deviations from the overall mean sum to zero.
      • With only 2 levels of A, the effect of A=2 is automatically minus the effect of A=1.
      • There is really only 1 A effect.
        o The df for factor A is 1.
• Degrees of freedom in general
  o If factor A has I levels.
    ▪ The A effects, deviations from average sum to 0.
    ▪ Once we know I-1 deviations from average, we know the last one automatically since they sum to zero.
    ▪ There are only I-1 independent A effects
      • A has I-1 df
    ▪ Similarly for factor B with J-1 levels
      • B has J-1 df
    ▪ Interaction degrees of freedom are the main effect df multiplied together.
      • The A*B notation actually comes from somewhere.
      • The A*B interaction has df = df_A * df_B
      • The A*B*C interaction has df = df_A * df_B * df_C
      • In a 2^k design all effects have 1 df
The data file for this analysis looks like

<table>
<thead>
<tr>
<th>Level of A</th>
<th>Level of B</th>
<th>Level of C</th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4 = X1*X2</th>
<th>X5 = X1*X3</th>
<th>X6= X2*X3</th>
<th>X7=X1<em>X2</em>X3</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>6.25</td>
</tr>
<tr>
<td>1</td>
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<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>7.00</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>4.75</td>
</tr>
<tr>
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<td>1</td>
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<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>4.50</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>7.00</td>
</tr>
<tr>
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<td>2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>10.00</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>4.50</td>
</tr>
<tr>
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<td>2</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>4.50</td>
</tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>15.50</td>
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<td>-1</td>
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<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>5.50</td>
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<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>10.00</td>
</tr>
<tr>
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<td>1</td>
<td>-1</td>
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<td>1</td>
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<td>-1</td>
<td>-1</td>
<td>16.00</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>6.00</td>
</tr>
<tr>
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<td>2</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5.75</td>
</tr>
</tbody>
</table>

- Again the notation A*B describes how the data column for the A*B effect is created.
  - By multiplying A*B where B is coded 1 and -1 for the two levels of B  
    - We are taking the A effect with B=Level 1 and subtracting the B effect with B=Level 2.
- The data sets from the authors have these -1 and 1 values for the main effects.
- For this example I have ignored the factor D in the dataset.
  - Factor D was not important, so this is not misleading.
  - If factor D had been important, leaving factor D out would be very misleading.
    - The variance estimate and standard errors would be much too big.
    - The error variance would included variability induced by varying levels of the important factor D.
Another Example:

- **Y** = Time to fill a container with syrup
- **A** = Brand: Pure maple syrup, Eggo, Hansen's
- **B** = Temperature: Room temp and refrigerated syrup

There is an obvious interaction between Brand and Temp.
- The time difference between Pure and Eggo syrup is less at room temp than refrigerated.
- The difference between refrigerated and room temp pouring times is greater for Hansen's than for pure maple syrup.

Ideally, we would like to be able to find more general conclusion for comparing Eggo and Hansen's syrups.
- The differences in pouring time are not the same, but we wouldn't really expect to see this happen.
- We would have anticipated larger differences in pouring times for the refrigerated, longer pouring times.
- Rather than consistent differences in pouring times, we would more likely anticipate similar relative changes, meaning similar ratios or percent changes.
- With similar ratios in the original time scale
  - We would see similar differences in the Log scale.
  - No interaction in the log scale.
Since we have similar differences in the Ln scale
  - The percentage or relative increase in pouring times for refrigerated syrups similar for all three syrups.
  - We are able to make more general conclusions.

Often, effects in factorial studies act multiplicatively rather than additively.

Suppose for example that in a 2x2 factorial
  - For factor A the effect of level 2 is to double the response.
  - For factor B the effect of level 2 is to triple the response.

\[
\begin{array}{c|c|c}
A & B=1 & B=2 \\
1 & 10 & 30 \\
2 & 20 & 60 \\
\end{array}
\]

- In this scale we have an obvious interaction.
- Going from B=1 to B=2 adds more to the response for A=2 than for A=1.
- But the effect of B is actually consistent.
- It’s just a multiplicative effect rather than adding fixed amounts.

By changing to a log scale, we can see this consistency.
- Equal ratios in the original scale correspond to equal differences in a log scale.
- Taking logs changes relative changes or ratios to absolute changes or differences.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>Y</th>
<th>Ratio</th>
<th>Log(Y)</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>10</td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>30</td>
<td>3:1</td>
<td>1.48</td>
<td>0.48</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>20</td>
<td></td>
<td>1.30</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>60</td>
<td>3:1</td>
<td>1.78</td>
<td>0.48</td>
</tr>
</tbody>
</table>

\[
\log(3Y) = \log(3) + \log(Y) = \log(Y) + 0.48
\]
- Also, often data have more consistent CV's across treatment groups with widely different means.
  - The standard deviations in the Ln scale correspond roughly to CV's in the original scale.
  - So if we have similar CV's in the original scale, we have similar variances in the Ln scale, one of the assumptions in the usual ANOVA model.
- If we find confidence intervals for differences in mean Ln(time)
  - Back-transforming the interval to the original scale gives us confidence intervals for ratio of pouring time in the time scale.

- Effect of Ignoring an Important Factor

![Syrup Ln Time](image)

- A t-test ignoring temperature for just Hansen's versus Eggo syrups.

<table>
<thead>
<tr>
<th>Brand</th>
<th>Method</th>
<th>Mean</th>
<th>95% CL Mean</th>
<th>Std Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eggo</td>
<td></td>
<td>3.0508</td>
<td>2.5183</td>
<td>3.5833</td>
</tr>
<tr>
<td>Hansen</td>
<td></td>
<td>3.4492</td>
<td>2.8931</td>
<td>4.0052</td>
</tr>
<tr>
<td>Diff (1-2)</td>
<td>Pooled</td>
<td>-0.3984</td>
<td>-1.1134</td>
<td>0.3167</td>
</tr>
<tr>
<td>Diff (1-2)</td>
<td>Satterthwaite</td>
<td>-0.3984</td>
<td>-1.1135</td>
<td>0.3168</td>
</tr>
</tbody>
</table>

| Method      | Variances   | DF    | t Value | Pr > |t| |
|-------------|-------------|-------|---------|------|---|
| Pooled      | Equal       | 18    | -1.17   | 0.2571 |
| Satterthwaite | Unequal    | 17.966 | -1.17   | 0.2571 |
- Factorial analysis with Temperature and Brand.

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brand</td>
<td>1</td>
<td>0.7936</td>
<td>0.7936</td>
<td>227.84</td>
<td>&lt;.0001</td>
</tr>
<tr>
<td>Temperature</td>
<td>1</td>
<td>10.3647</td>
<td>10.3647</td>
<td>2975.78</td>
<td>&lt;.0001</td>
</tr>
<tr>
<td>Brand*Temperature</td>
<td>1</td>
<td>0.0050</td>
<td>0.0050</td>
<td>1.44</td>
<td>0.2472</td>
</tr>
</tbody>
</table>

- Which analysis gives answers consistent with the plotted data?
  - **Always** confirm your results by plotting the data!
  - What went wrong with the incorrect analysis?
Residuals

The fitted value for each group is the mean for that treatment group

\[ \hat{y}_{ijk} = \bar{y}_{ij} \]

\[ \hat{y}_{ijk} = k^{th} \]

The residuals are

\[ e = y - \hat{y} \]

\[ e_{ijk} = y_{ijk} - \hat{y}_{ijk} = y_{ijk} - \bar{y}_{ij} \]

Plotting residuals

In addition to the initial plot, plot residuals to check for

- Equal variance
- Drift over time
- Normality

Plot residuals

- Versus predicted values (treatment group means)
- Versus run order from 1 to I·J·n

Also plot residuals versus

- A levels
- B levels

For only two factors, the plot of all original data points shows pretty well how for example variances change with the factors. For factorial studies with more than 2 factors, we can’t plot all original data points one plot, so it becomes more essential to plot residuals versus the factors.

Normal plot or plots

- Residuals.
  - For all residuals together if variances appear similar throughout.
  - With enough data in each group, we could also look at normal plots for each treatment group on the same plot.
Possibly a bit of increasing variance.

Possibly a bit of increasing variance.
Possibly a bit of increasing variance.

No apparent trends.

Residuals appear normal
Tests and Confidence Intervals

Section 8.1 Two-way Factorial

Bricks Example 5 on Page 467

<table>
<thead>
<tr>
<th>Heat Treat</th>
<th>% water</th>
<th>Brick Strength</th>
<th>( \bar{x} )</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slow</td>
<td>17</td>
<td>4911, 5998, 5676</td>
<td>5528</td>
<td>558</td>
</tr>
<tr>
<td>Slow</td>
<td>19</td>
<td>4387, 5388, 5007</td>
<td>4927</td>
<td>505</td>
</tr>
<tr>
<td>Fast</td>
<td>17</td>
<td>3824, 3140, 3502</td>
<td>3489</td>
<td>312</td>
</tr>
<tr>
<td>Fast</td>
<td>19</td>
<td>4768, 3672, 3242</td>
<td>3894</td>
<td>787</td>
</tr>
</tbody>
</table>

As with t-tests we estimate the variance, \( \sigma \), with the pooled variance, \( S_p^2 \).

\[
SSE = SSError = \sum \sum \sum \left( y_{ijk} - \bar{y}_{ij} \right)^2 = \sum \sum \sum \left( e_{ijk} \right)^2 = \sum \sum \sum \left( residual_{ijk} \right)^2
\]

\[
MSE = MSError = \frac{\sum \sum \sum \left( n_{ij} - 1 \right) S_{ij}^2}{\sum \sum \left( n_{ij} - 1 \right)} = S_p^2
\]

With equal n's, \( S_p^2 = \text{unweighted average } S_{ij}^2 \).

\( y_{ijk} \) is the kth value in treatment group with A=i and B=j.

\[
S_p^2 = \frac{558^2 + 505^2 + 312^2 + 787^2}{4} = 325,646
\]

Heat main effect

\[
\bar{Y}_{slow} = \frac{1}{2} (5528) + \frac{1}{2} (4927) = 5227.5
\]

\[
\bar{Y}_{fast} = \frac{1}{2} (3489) + \frac{1}{2} (3894) = 3691.5
\]

\[
\hat{L} = \frac{1}{2} \bar{Y}_{11} + \frac{1}{2} \bar{Y}_{12} - \frac{1}{2} \bar{Y}_{21} - \frac{1}{2} \bar{Y}_{22} = 1536
\]

\[
\sqrt{V(\hat{L})} = \sqrt{\frac{4(1/2)^2 S_p^2}{3}} = \sqrt{\frac{S_p^2}{3}}
\]

\[
= 329.6
\]
With equal n's this can be simplified

\[ \bar{Y}_{\text{slow}} = \text{average of 6 values} \]
\[ \bar{Y}_{\text{fast}} = \text{average of 6 values} \]

\[ SE = \sqrt{\hat{V}(\bar{Y}_{\text{slow}} - \bar{Y}_{\text{fast}})} = \sqrt{\frac{S_p^2}{6} + \frac{S_p^2}{6}} = \sqrt{\frac{S_p^2}{3}} \]

95% CI  \hspace{1cm} 1536 \pm 2.306(329.6)  
\hspace{1cm} 1536 \pm 760 
\hspace{1cm} 776 \text{ to } 2296 

With equal numbers of values in each group, the comparison of the heat levels is equivalent to

\[ \bar{Y}_{\text{slow}} = \text{average of 6 values} \]
\[ \bar{Y}_{\text{fast}} = \text{average of 6 values} \]

\[ \sqrt{\hat{V}(\bar{Y}_{\text{slow}} - \bar{Y}_{\text{fast}})} = \sqrt{\frac{S_p^2}{6} + \frac{S_p^2}{6}} \]

To test Ho : Heat main effect = 0 \hspace{1cm} Ha : Heat main effect \neq 0

\[ t = \frac{1536}{329.6} = 4.66 \]

\[ 2 \times 0.005 < p-value < 2 \times 0.001 \]
\[ 0.001 < p-value < 0.002 \]
### ANOVA Table

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heat</td>
<td>1</td>
<td>7080960</td>
<td>7080960</td>
<td>21.74</td>
<td>0.002</td>
</tr>
<tr>
<td>Water</td>
<td>1</td>
<td>28616</td>
<td>28616</td>
<td>0.09</td>
<td>0.774</td>
</tr>
<tr>
<td>Heat*Water</td>
<td>1</td>
<td>760033</td>
<td>760033</td>
<td>2.33</td>
<td>0.165</td>
</tr>
<tr>
<td>Error</td>
<td>8</td>
<td>2605169</td>
<td>325646</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>10474779</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The t test value for the heat effect was $4.66 = \sqrt{21.74}$.

This was not a coincidence. An F with 1 df in the numerator is equivalent to $t^2$

$$F_{1,df} = t_{df}^2$$

![Bricks](image)

- Just as in the regression ANOVA tables, the Mean Square Residual or Mean Square Error is our estimate of $\sigma^2$.
- For factorial studies with replication, MS Error = pooled variance.