Discrete Random Variables

- Expected value of X = E(X) = \( \mu = \sum xP(x) = \sum xf(x) \)
  - Mean = Expected value of X
  - Weighted average of potential x values = center of gravity.
- Variance of X = Var(X) = E((X-\( \mu \))^2) = \( \sigma^2 = \sum (x-\mu)^2 P(x) = \sum (x-\mu)^2 f(x) \)
  - Variance = Expected squared deviation from average
  - Weighted average of \((x-\mu)^2\)
- \( \mu \) and \( \sigma^2 \) are the mean and variance of the population of values if we repeated the experiment an infinite number of times.

- Binomial
  - X = Number of times "S = Something" happens in n independent trials with constant probability p = P(S).
    - In section 5.1 S stands for "success" on each trial
  - Number of trials is fixed in advance = fixed n
  - Number of times S happens = random X
  - \( P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \)
  - E(X) = \mu = n*p
    - Toss a coin 100 times. X = # heads. E(X) = \mu = 100(0.5) = 50
  - Var(X) = n*p(1-p)
- Geometric
  - X = number of tries until first "S = something" happens in independent trials with constant P(S) = p
  - Number of trials = random X
  - Number of times S happens = fixed as 1
  - Memoryless property: X = tosses until first 6 on a die. Having tossed a die 5 times without a six, our expected number of further tosses is the same as the expected number of tosses when we started.
  - \( P(x) = (1-p)^{x-1} p \)
  - E(X) = \mu = 1/p
    - X = number of tosses of a die until first 5. E(X) = \frac{1}{1/6} = 6
  - Var(X) = (1-p)/p^2
- Poisson
  - X = number of totally haphazard, independent events in span of time or space
    - Constant rate of events.
    - Each event is completely independent.
      - An event in one time period says nothing about the chance of the event in another time period
Knowing that the previous ship arrived two hours or two minutes ago tells us nothing about when the next ship will come in.

Memoryless property: Having waited an hour with no ships arriving, our expected waiting time is the same as the expected waiting time when we showed up.

\[ P(x) = \frac{e^{-\lambda} \lambda^x}{x!} \]

- Expected number of bubbles in sheets of glass
- Expected number of particles emitted from radioactive substance

\[ \text{Var}(X) = \lambda \]

\[ P(E_1 \text{ and } E_2) = P(E_1) \times P(E_2) \] for independent events

- \( S_1 = 6 \) on die #1. \( S_2 = 6 \) on die #2. \( P(S_1 \text{ and } S_1) = P(S_1) \times P(S_2) = (1/6) \times (1/6) \)

\[ P(E_2 \text{ given } E_1) \text{ is written } P(E_2 | E_1) \]

- \( E_1 \text{ and } E_2 \) are independent if \( P(E_2 | E_1) = P(E_1) \)
- \( P(S_1 = \text{ace first} \text{ and } S_2 = \text{ace second}) = P(A_1 \text{ and } A_2) = P(A_1) \times P(A_2 | A_1) = (4/52) \times (3/51) \)

\[ P(\text{not } E) = 1 - P(E) \quad P(E) = 1 - P(\text{not } E) \]

- \( P(\text{at least one 6 in three tosses of die}) = 1 - P(\text{not at least one 6}) = 1 - P(0 \text{ sixes}) \)
  \[ = 1 - P(\text{Not 6}_1 \text{ and Not 6}_2 \text{ and Not 6}_3) = 1 - P(N_1) \times P(N_2) \times P(N_3) \]
  \[ = 1 - \frac{5}{6}^3 \]

- \( P(\text{at least 2 defects}) = 1 - P(\text{0 defect or 1 defect}) = 1 - (P(0 \text{ defects}) + P(1 \text{ defect}) ) \]

\[ P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) \] as long as

- There are no outcomes in common in \( E_1, E_2 \)
- Having \( E_1 \) happen excludes the possibility of \( E_2 \) happening: mutually exclusive
- \( P(E_1 \text{ and } E_2) = 0 \)
- \( P(\text{Ace first or King first}) = P(\text{Ace first}) + P(\text{King first}) = 4/52 + 4/52 \)

\[ P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2) \] for any \( E_1, E_2 \)

- \( P(E_1) + P(E_2) \) counts outcomes in common twice, so subtract these once.
- \( A_1 = \text{Ace on first card} \quad H_1 = \text{Heart on first card} \)
  \[ P(A_1) + P(H_1) \text{ counts ace of hearts twice.} \]
  \[ P(A_1 \text{ or } A_2) = P(A_1) + P(A_2) - P(A_1 \text{ and } A_2) = 4/52 + 12/52 - 1/52 = 16/52 \]

\[ P(E_1 \text{ or } E_2 \text{ or } E_3) = 1 - P(\text{not } E_1 \text{ or } E_2 \text{ or } E_3) = 1 - P(\text{not(E1)} \text{ and not(E2)} \text{ and not(E3)}) \]

For independent events \( P(E_1 \text{ or } E_2 \text{ or } E_3) = 1 - [1-P(E_1)] \times [1-P(E_2)] \times [1-P(E_3)] \)

- Toss a die 4 times. \( S = \text{roll a 6} \).
  \[ P(S_1 \text{ or } S_2 \text{ or } S_3 \text{ or } S_4) = 1 - P(\text{Not } S_1 \text{ and not } S_2 \text{ and not } S_3 \text{ and not } S_4) \]
  \[ = 1 - P(\text{not } S_1) \times P(\text{not } S_2) \times P(\text{not } S_3) \times P(\text{not } S_4) = 1 - \left( \frac{5}{6} \right)^4 \]

- Alternatively, \( X = \text{number of 6's} \text{ - Binomial} \).
  \[ P(S_1 \text{ or } S_2 \text{ or } S_3 \text{ or } S_4) = P(X \neq 0) = 1 - P(0) = 1 - (1-p)^4 = 1 - (5/6)^4 \]