5.2.3 Normal Distribution

The density function is

\[ f(x) = \frac{1}{\sqrt{2\pi\sigma}} \times e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} = \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{1}{2}z^2} \]

\[ z = \frac{x-\mu}{\sigma} = \text{standard deviation from average} \]

For \( \mu = 0 \) and \( \sigma = 1 \) the normal distribution is called Standard Normal

\[ P(a \leq X \leq b) = \text{Area} \]

Table B.3 gives probabilities for standard normal
(Numerical Integration - No formula for \( \int_a^b \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \, dz \))

\[ \Phi(z) = F(z) = P(Z \leq z) \]
From Table B.3

\[ P(z \leq 1.5) = 0.9332 \]

\[ P(Z \geq 1.5) = 1 - 0.9332 = 0.0668 \]

If \( X \sim N(\mu, \sigma^2) \), then \( Z = \frac{X - \mu}{\sigma} \sim N(0, 1) \)

\[ E(\frac{X - \mu}{\sigma}) = \frac{1}{\sigma} \times E(X - \mu) = \frac{1}{\sigma}[\mu - \mu] = 0 \]

\[ \text{St. Dev}(\frac{X - \mu}{\sigma}) = \frac{1}{\sigma} \times \text{St. Dev}(X) = \frac{1}{\sigma} \times \sigma = 1 \]

Example: \( X = \) breaking strength, \( \mu = 9000 \), and \( \sigma = 1000 \)

\[ P(X < 7500) = P(\frac{X - \mu}{\sigma} < \frac{7500 - 9000}{1000}) = P(Z < -1.5) = 0.0668 \]

(Same as \( P(Z > 1.51) \))
\[ P(7000 < x < 8000) = P\left(\frac{7000 - 9000}{1000} < Z < \frac{8000 - 9000}{1000}\right) = P(-2 < Z < -1) \]

0.1587 - 0.0228 = 0.1359

If \( \ln(X) \) is normal, then \( X \) has a lognormal distribution.

If \( Y = \ln(X) \sim N(\mu, \sigma^2) \), then the coefficient of variation of \( X \) is approximately \( \sigma \).

\[
CV = \frac{\sigma}{\mu}
\]

Deviations in log scale correspond to relative deviations in original scale.

Standard normal quantile function

Notation: \( Q_z(p) = 100 \times p^{th} \) percentile for standard normal

1.645 = 90^{th} percentile

90\% of values in a normal pop’n are less than 1.645 St. Dev’s above \( \mu \).
5.2.4 Exponential Distribution

Ships arrive 2 ships/hour. The number of ships arriving in 1 hour is Poisson($\lambda = 2$)

$X = \text{Time until the next ship.}$

$P(X \leq x) = P(0 \text{ ships by the time } x)$

$\# \text{ of ships in time } x \sim \text{Poisson}(\lambda = 2x)$

$P(0 \text{ ships}) = e^{-2x}(2x)^0/0! = e^{-2x}$

$F(x) = P(X \leq x) = 1 - P(X \geq x) = 1 - e^{-2x}$

$f(x) = \text{density} = \frac{d}{dx} F(x) = 2e^{-2x}$

$E(X) = \int_{0}^{\infty} x2e^{-2x}dx = 1/2$

An exponential distribution with mean $\mu = \alpha$ has density function

$f(x) = \frac{1}{\alpha}e^{-\frac{x}{\alpha}}, \quad x > 0$

$F(x) = 1 - e^{-\frac{x}{\alpha}}$

As above with $E(X) = \alpha = \frac{1}{2}$

$f(x) = 2e^{-2x} \quad \mu = \frac{1}{1/2} = 2$

The variance is $\sigma^2 = \alpha^2 \quad \sigma = \alpha$
Memoryless property: If we have already waited $H$ hours and haven’t seen a ship, our expected waiting time is $H + \frac{1}{2}$. Our expected waiting time is like starting all over again. The prob’y of a ship showing up in the next 5 minutes is the same as when we started.

The force-of-mortality function is (p.760)

$$h(t) = \frac{f(t)}{P(T > t)} = \frac{f(t)}{1 - F(t)}$$

$h(t) dt$ is the probability of dying in time $t$ to $t + dt$ if we are still living at $t$.

For exponential distributions

$$h(t) = \frac{1}{\alpha} e^{-x/\alpha}$$

a constant force-of-mortality

The geometric distribution was also memoryless.

$X = \text{time to 1st success.}$

The expected tosses to next head doesn’t depend on how long we have been tossing without getting a head.

Lifetimes of glasses in a restaurant might be exponential. Motor lifetimes or people’s life times are not. Sometimes lifetimes can be modeled with a lognormal distribution where $\log(X) \sim \text{Normal}$
5.2.5 Weibull Distribution

Very commonly lifetimes of motors etc. are modeled with Weibull distributions. A Weibull distribution is a generalization of an exponential distribution, an exponential raised to some power.

If \( X \) is Weibull, then

\[
\left( \frac{X}{\alpha} \right)^{\beta} \sim \text{Exponential with mean 1}
\]

or equivalently a Weibull random variable, \( X \), is an exponential random variable raised to the \( \frac{1}{\beta} \) power

\[
F(x) = P(X \leq x)) = P \left( \left( \frac{X}{\alpha} \right)^{\beta} < \left( \frac{x}{\alpha} \right)^{\beta} \right) = 1 - e^{-(x/\alpha)^{\beta}}
\]

\[
f(x) = \frac{d}{dx} F(x) = \frac{\beta}{\alpha} \left( \frac{x}{\alpha} \right)^{\beta-1} e^{-(x/\alpha)^{\beta}}
\]

\[
\mu = E(X) = \alpha \Gamma \left( 1 + \frac{1}{\beta} \right)
\]

Gamma function \( \Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt \)

\( \Gamma(n) = (n - 1)! \)

- For \( \beta < 1 \) the force-of-mortality is a decreasing function, for example, a product break in period.

- \( \beta = 1 \) Constant force-of-mortality. Exponential distribution

- \( \beta > 1 \) Increasing force-of-mortality, for example, a product that wears out.