Sylwia CICHACZ, Jakub PRZYBYŁO
and
Mariusz WOŹNIAK

Irregular edge-colorings
of sums of cycles
of even lengths

Preprint Nr MD 023
(otrzymany dnia 2 grudnia 2006)
Redaktorami serii preprintów Matematyka Dyskretna są:
Wit FORYŚ,
prowadzący seminarium Słowa, słowa, słowa...
w Instytucie Informatyki UJ
oraz
Mariusz WOŹNIAK,
prowadzący seminarium Matematyka Dyskretna - Teoria Grafów
na Wydziale Matematyki Stosowanej AGH.
Irregular edge-colorings of sums of cycles of even lengths

Sylwia Cichacz, Jakub Przybyło*, Mariusz Woźniak
AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland

December 2, 2006

Abstract

P. Wittmann showed that for the irregular coloring number \( c(G) \) of a simple 2-regular graph of order \( n \) the inequality \( c(G) \leq \sqrt{2n} + O(1) \) holds. We determine the exact value of this number in the case when 2-regular graph consists of cycles of even lengths. For this purpose we consider decompositions of several classes of graphs.

1 Introduction

Consider a simple (without loops and multiple edges) nondirected graph \( G \). Let \( C \) be a color set, \( w: E(G) \to C \) an edge coloring and let \( S(v) \) denote a multiset of colors of all edges incident with a vertex \( v \) in \( G \). A coloring \( w \) is said to be irregular if for any two distinct vertices \( u, v \) the corresponding multisets hold \( S(u) \neq S(v) \). We ask for the minimal number of necessary colors to obtain an irregular edge coloring and we call it an irregular coloring number. Moreover, we denote by \( c(G) \) the irregular coloring number of a given graph \( G \).

Such a number has also another interesting interpretation, as a variant of irregular weighting of edges of a graph \( G \) with positive integers. Namely, \( c(G) \) can be consider as the minimal cardinality of such a subset of \( \mathbb{N} \) that allows us to distinguish all vertices of \( G \) by the sums of labels of edges adjacent to them, see [11] for details.

The irregular coloring number is, for instance, determined to be equal to three for graphs \( K_n \) and \( K_{n,n} \). Other results for connected graphs, such as

*Corresponding author. E-mail: przybylo@wms.mat.agh.edu.pl
multipartite graphs, are also known, see [11]. However, since we are interested in global distinguishing of vertices (not only neighboring ones), these results do not yield an instant generalization for nonconnected graphs, even if these graphs are disjoint unions of the connected graphs mentioned above. Already 2-regular graphs turned out to be problematic. A representative of such a family is a disjoint union of cycles and can be denoted as \( G = C_{a_1} \cup \ldots \cup C_{a_p} \), where \( C_i \) is a cycle of length \( i \). The following upper bound was first established by M. Aigner et al.

**Theorem 1 ([1])**  
Let \( G = C_{a_1} \cup \ldots \cup C_{a_p} \) be a simple 2-regular graph of order \( n = \sum_{i=1}^{p} a_i \). Then

\[
c(G) \leq \sqrt{8n} + O(1).
\]

It was then improved by P. Wittmann.

**Theorem 2 ([11])**  
Let \( G = C_{a_1} \cup \ldots \cup C_{a_p} \) be a simple 2-regular graph of order \( n = \sum_{i=1}^{p} a_i \). Then

\[
c(G) \leq \sqrt{2n} + O(1).
\]

This result is best possible except for an additive constant term. In this paper, which is a continuation of our reasonings from [7], we determine the exact value of the irregular coloring number for 2-regular graphs consisting of cycles of even lengths.

### 2 Coloring and decomposition

Similarly as authors mentioned above, we use the following correspondence between an irregular edge coloring \( w \) of a 2-regular graph \( G = C_{a_1} \cup \ldots \cup C_{a_p} \) with \( r \) colors and an (edge-disjoint) packing of (connected) Eulerian subgraphs into the graph \( M_r \), where \( M_r \) is a complete graph \( K_r \) with a loop at each vertex added (Though \( M_r \) contains loops, we shall call it a graph).

First identify the vertices of \( M_r \) with the colors of \( w \). Now choose an arbitrary \( C_{a_i} \) and for any two colors appearing in some \( S(u) \) of \( C_{a_i} \), draw an edge or a loop between the corresponding vertices of \( M_r \) (Notice that each multiset \( S(u) \) consists of just two colors and that we draw a loop in \( M_r \) only if these colors are the same). Since \( S(u) \neq S(v) \) for any two distinct vertices of \( C_{a_i} \), we never draw an edge of \( M_r \) twice. Moreover, as in the following example (see Figure 1), traversing \( C_{a_i} \) yields a corresponding Eulerian subgraph \( G_{a_i} \) of size \( a_i \) in \( M_r \). Since \( w \) is an irregular edge coloring of the graph
Let \( G = C_{a_1} \cup \ldots \cup C_{a_p} \), we obtain edge-disjoint Eulerian subgraphs of sizes \( a_1, \ldots, a_p \) in \( M_r \). Clearly, this procedure works the other way around as well, hence we have reduced the problem of irregular edge coloring to the following packing problem:

Let \( G = C_{a_1}, \ldots, C_{a_p} \), then \( c(G) \) is the smallest number \( r \) such that we can (edge-disjointly) pack Eulerian subgraphs of sizes \( a_1, \ldots, a_p \) into \( M_r \).

We shall call a (connected) Eulerian graph (or subgraph) of size \( n \), a closed trail of length \( n \). Notice that such a closed trail can be identified with any sequence \( (v_1, v_2, \ldots, v_{n+1}) \) of its, not necessarily distinct, vertices such that \( v_i v_{i+1} \) are its distinct edges for \( i = 1, \ldots, n \), hence \( v_1 = v_{n+1} \). Moreover, given two edge-disjoint closed trails \( A_1, A_2 \) which are not disjoint on vertices, we shall write \( A_1 A_2 \) for their union, which is a closed trail as well. Notice also that if \( A_1, \ldots, A_p \) are edge-disjoint closed trails in \( M_n \), their union forms an Eulerian subgraph (or a union of Eulerian subgraphs) of \( M_n \). Therefore, instead of the problem of packing, we consider a problem of decomposing into closed trails of even lengths of \( L_n \), which is defined to be a maximal, in terms of size, Eulerian subgraph of \( M_n \) with even number of edges.

Therefore, we introduce the following definitions. A sequence \( \tau = (a_1, \ldots, a_p) \) of integers is called admissible for \( G \) if its elements add up to \( \|G\| \), the size of the graph \( G \), and \( a_i \geq 3 \) for \( i = 1, \ldots, p \). Moreover, if \( G \) can be (edge-disjointly) decomposed into closed trails \( A_1, A_2, \ldots, A_p \) of lengths \( a_1, a_2, \ldots, a_p \), respectively, then \( \tau \) is called realizable in \( G \) and the sequence \( (A_1, A_2, \ldots, A_p) \) is said to be a \( G \)-realization of \( \tau \) or a realization of \( \tau \) in \( G \). Since we are interested in unions of cycles of even lengths, we exclusively investigate sequences consisting of numbers divisible by two and we call them even sequences. Moreover, we say that a graph \( G \) is arbitrarily decomposable into closed trails (of even lengths) if each (even) admissible for \( G \) sequence is also realizable in this graph.

Our main aim will be to show that \( L_n \) (and \( L_n \) with a pair of loops removed) is arbitrarily decomposable into closed trails of even lengths for
(almost) every \( n \). In other words, we show that we can (edge-disjointly) pack as many closed trails of even lengths into \( M_n \) as it is possible taking into account the necessary conditions, i.e. the size of the graph and the fact that these closed trails are its Eulerian subgraphs, see Theorem 14. Such an optimal solution of this problem results in determining \( c(G) \) for 2–regular graphs consisting of cycles of even lengths, see Theorem 15.

It is also worth mentioning here, that similar problems concerning decompositions, but in the case of the complete graphs, were investigated by P.N. Balister, whose best known result is as follows.

**Theorem 3** ([2]) Let the sum \( \sum_{i=1}^{p} a_i, a_i \geq 3 \) be equal \( \binom{n}{2} \) when \( n \) is odd and \( \left( \frac{n}{2} \right)^2 - n - 2 \leq \sum_{i=1}^{p} a_i \leq \left( \frac{n}{2} \right)^2 - \frac{n}{2} \) when \( n \) is even. Then we can decompose some subgraph of \( K_n \) into closed trails of lengths \( a_1, \ldots, a_p \).

This theorem enabled to establish the value of a variant of irregular coloring number for 2-regular graphs in the case when we assume this coloring to be proper. Also directed graphs were discussed by the same author, see [3]. Other related problems appear in [5] and [10]. In our proof, the most useful will be the following result of M.Horňák and M. Woźniak.

**Theorem 4** ([10]) If \( a, b \) are positive even integers, then if \( \sum_{i=1}^{p} a_i = a \cdot b \) and there is a closed trail of length \( a_i \) in \( K_{a,b} \) (for all \( i \in \{1, \ldots, p\} \)), then \( K_{a,b} \) can be (edge-disjointly) decomposed into closed trails \( A_1, A_2, \ldots, A_p \) of lengths \( a_1, a_2, \ldots, a_p \), respectively.

Let us observe that \( K_{2,b} \) contains only closed trails of lengths \( 4i \), where \( i = 1, 2, \ldots, \frac{b}{2} \), whereas \( K_{a,b} \), for \( a, b \geq 4 \), contains closed trails of lengths \( 2j \), where \( j = 2, 3, \ldots, \frac{ab-4}{2}, \frac{ab}{2} \), see [10].

### 3 Decomposition of \( L_m^m \)

Notice first that \( L_n \) for even \( n \) is a complete graph \( K_n \) with a single loop at each vertex added and minus 1-factor. Let \( L'_n \) (\( n \) - even) denote a graph \( L_n \) with two loops at nonadjacent vertices removed. In [7] we proved the following theorem.

**Theorem 5** Graphs \( L_n \) and \( L'_n \) with even \( n \) are arbitrarily decomposable into closed trails of even lengths, unless \( n = 4 \) (and \( \tau = (4,4) \)).

Observe in turn, that if we take an odd \( n \) into account, then \( L_n \) is equal to \( M_n \) if \( n \equiv 3(\text{mod } 4) \) and is equal to \( M_n \) with one loop removed otherwise. Let this time \( L'_n \) denote a graph \( L_n \) with an arbitrary pair of loops removed.
For technical reasons, to show that graphs $L_n$ and $L'_n$ (n - odd) are arbitrarily decomposable into closed trails of even lengths, we first have to show that such a statement remains true for another family of graphs.

Let then for even integers $m, n$, with $0 \leq m \leq n$, $\mathcal{L}_n^m$ stand for a family of all graphs we may receive by adding single loops at $m$ vertices of $K_n$ minus 1-factor. Notice that $L_n \in \mathcal{L}_n^0$ and is actually (up to isomorphism) the only representative of this family as well as $K_n$ minus 1-factor is a single representative of $\mathcal{L}_n^0$. This is however not the case if $2 \leq m \leq n - 2$, therefore, from now on, by $L_n^m$ we will usually mean an arbitrary representative of the family $\mathcal{L}_n^m$ if nothing else is stated. This section is dedicated to proving the following theorem.

**Theorem 6** Let $m, n$ be even numbers, where $0 \leq m \leq n$. A graph $L_n^m \in \mathcal{L}_n^m$ is arbitrarily decomposable into closed trails of even lengths, unless $n = 4$ and $m = 4$.

Let $x$ be a vertex of $L_n^m$. The only nonadjacent to $x$ vertex of $L_n^m$ we shall denote by $x'$, hence $(x')' = x$. We say that $x$ is of type one if neither $x$ nor $x'$ has a loop, whereas it is of type three if there is a loop at both of them. Analogously, $x$ is of tape two or four if only $x'$ or only $x$, respectively, has a loop. We shall write $t(x) = i$, where $i = 1, 3, 2, 4$, respectively, see Figure 2. Notice that the type of $x'$ is the consequence of the type of $x$, since either $t(x) = 1 = t(x')$ or $t(x) = 3 = t(x')$ or $\{t(x), t(x')\} = \{2, 4\}$.

![Figure 2: Types of a vertex $x$.](image)

Observe that if $m = 0$, then Theorem 6 is true by Theorem 3, whereas for $m = n$ we are done by Theorem 5. Therefore, we can restrict our reasonings to the case $2 \leq m \leq n - 2$. But then, a graph $L_n^m$ contains an induced subgraph $H_1$ or $H_2$ from Figure 3. It is because then there exist vertices $x, x', y, y' \in V(L_n^m)$, such that either $t(x) = t(y) = 2$ or $t(x) = 1$ and $t(y) = 3$. Notice that the graphs $H_1$ and $H_2$ are actually the only nonisomorphic representatives of the family $\mathcal{L}_n^2$. Therefore, there exist $L_3^2 \in \mathcal{L}_n^2$ and $L_n^{-2} \in \mathcal{L}_n^{-2}$ such that $L_n^m = (L_3^2, (K_{4n-4}), L_n^{-2})$. Let us then denote by $\mathcal{R}_n$ a family
of graphs of the form $L_2^2, K_{4,n-4}$, where a vertex set of $L_2^2 \in L_4^2$ coincide with a partition set of size 4 of $K_{4,n-4}$. An arbitrary representative of this family we shall denote by $R_n$, see Figure 4. The basic idea of our proof is to consider

a graph $G = L_n^m$ as a union $G',G''$ of the two graphs and given an admissible for $G$ sequence $\tau = (a_1, \ldots, a_p)$, divide it into two sequences $\tau' = (a_1, \ldots, a_i)$, $\tau'' = (a_{i+1}, \ldots, a_p)$ admissible for $G'$, $G''$, respectively, and decompose these two graphs separately. Therefore, if $2 \leq m \leq n-2$, we can take $G' = R_n$ and $G'' = L_{n-4}^{m-2}$. If additionally $m \geq 4$, then we can assume $G' = R_n, L_2$ and $G'' = L_{n-4}^{m-4}$, where by $R_n, L_2$ we mean one of the members of the family $\mathcal{R}_n$ with two loops at two vertices from the partition set of size $n-4$ of $K_{4,n-4}$ added. Then we decompose $G''$ by induction and $G'$ by one of the two lemmas bellow. It is however obvious, we can not always simply divide $\tau$ into $\tau'$ and $\tau''$ as described above. Therefore, we split $a_i = a'_i + a''_i$ at times and search for realizations of $\tau'_i = (a_1, \ldots, a_{i-1}, a'_i)$ and $\tau''_i = (a''_i, a_{i+1}, \ldots, a_p)$ in $G'$ and $G''$, respectively, and finally glue together closed trails of lengths $a'_i$ and $a''_i$ to form the one of length $a_i$. This is, in turn, possible only if the closed trail of length $a'_i$ meets at least one vertex from the partition set of size $n-4$ of $K_{4,n-4}$, see Figure 4.
From now on, we shall write \(a_1^{r_1} \cdot a_2^{r_2} \cdot \ldots \cdot a_s^{r_s}\) instead of the sequence \((a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_s, \ldots, a_s)\) for short. Moreover, if \(r_i = 1\) for some \(i\), we shall omit writing \(r_i\) in this shortened notation.

**Lemma 7** A graph \(R_n \in \mathcal{R}_n\), with \(n \geq 8\), is arbitrarily decomposable into closed trails of even lengths.

**Proof.** Let \(\tau = (a_1, \ldots, a_p)\) be an even admissible sequence for \(R_n\). Since \(\|R_n\| \equiv 2 \pmod{4}\), we may assume \(a_1 \equiv 2 \pmod{4}\) and find a realization of \(\tau_1 = (a_1 - 6, a_2, \ldots, a_p)\) in \(K_{4,n-4}\) by Theorem 4 (in particular we may receive \(a_1 - 6 = 0\)). Then, by gluing together a closed trail of length \(a_1 - 6\) with \(L_4^2\), we receive a realization of \(\tau\) in \(G\). \(\blacksquare\)

Notice that if we want to be certain we can choose this realization of \(\tau\) in such a way, that a closed trail of a given length, say \(a_1\), is a subset of \(K_{4,n-4}\). It is enough to assume either \(a_j \notin \{4,8\}\) or \(a_{j_1}, a_{j_2} = 8\) for some \(j, j_1, j_2 > 1\). It is obvious by the proof above when \(a_j = 6\) or \(a_j \geq 10\). It is enough to exchange \(a_1\) with \(a_j\). In the second case, if for instance \(a_2 = a_3 = 8\), we find a realization of \(\tau_2 = (a_1, a_2 - 2, a_3 - 4, a_4, \ldots, a_p)\) in \(K_{4,n-4}\) and glue together (possibly after a permutation of vertices of \(K_{4,n-4}\)) a closed trail of length \(a_2 - 2\) with the two loops from \(L_4^2\) and the one of length \(a_3 - 4\) with the rest of \(L_4^2\).

It is also obvious that in these two cases a closed trail of length \(a_1\) has at least one vertex in the partition set of size \(n - 4\) of \(K_{4,n-4}\). However, since \(\|L_4^2\| = 6\), it is also the case if \(a_1 \geq 8\).

**Lemma 8** If \(\tau = (a_1, \ldots, a_p)\) is an even admissible sequence for \(G = R_n, L_2\), where \(R_n \in \mathcal{R}_n\) and \(n \geq 8\), then it is also \(G\)-realizable, unless \(\tau = 4^r\) for some \(r\).

**Proof.** If one of the elements of \(\tau\), say \(a_1\), is not smaller than 12, then we find a realization of \(\tau_1 = (a_1 - 8, a_2, \ldots, a_p)\) in \(K_{4,n-4}\) by Theorem 4 and then glue together a closed trail of length \(a_1 - 8\) with \(L_4^2\) and \(L_2\). Analogously, if there are at least two elements of \(\tau\), say \(a_1, a_2\), not divisible by four, then we find a realization of \(\tau_2 = (a_1 - 6, a_2 - 2, a_3, \ldots, a_p)\) in \(K_{4,n-4}\) and glue together a closed trail of length \(a_1 - 2\) with the two loops \(L_2\) and a closed trail of length \(a_2 - 6\) with \(L_4^2\). In both cases some permutations of vertices of \(K_{4,n-4}\) may be necessary.

Therefore, since \(\|G\| \equiv 0 \pmod{4}\), we can assume \(a_j \in \{4,8\}\) for all \(j\) and \(a_1 = 8\) (because \(\tau \neq 4^r\)). Then we find a realization of \(\tau_3 = \tau_2 \cdot \tau_2 \cdot \ldots \cdot \tau_2\) (where \(\tau_2\) is the \(4\)-adic realization of \(\tau\)) with the two loops from \(L_4^2\) and the one of length \(a_3 - 4\) with the rest of \(L_4^2\).
Let \( G \parallel x, y, z \) exist. Let first \( G = 4\tau \) be a programme we created. Assume then \( \tau = 0 \) and \( m = 2 \). Let \( \tau = 0 \) and \( m = 2 \). Then, since \( 4p + 6q + 8r = \|G\| \) in the following two cases:

1. \( G = K_{a,b} \) if \( a \geq 4 \), \( b \geq 6 \) and \( a, b \) are even.
2. \( G = K_{a,a} \) minus 1-factor if \( a \) is odd.

**Proof of Theorem 6.** The cases for \( n \leq 10 \) has been analyzed by a computer programme we created. Assume then \( n \geq 12 \) and let us argue by induction on \( n \).

Let \( G = L_n^m \) be an arbitrary representative of a family \( L_n^m \) and let \( \tau = (a_1, \ldots, a_p) \) be an even admissible sequence for \( G \). Since the cases \( m = 0 \) and \( m = n \) are the consequence of Theorems 3 and 5, we may assume \( 2 \leq m \leq n - 2 \). Let \( s_i = a_1 + a_2 + \ldots + a_i \) for each \( i \).

Let first \( \tau = 4p \). If there exists a vertex \( x \in V(G) \) such that \( t(x) = 1 \), then \( G = K_{2,n-2}.L_{n-2}^m \) for some \( L_{n-2}^m \in L_{n-2}^m \), where one of the partition sets of \( K_{2,n-2} \) equals \( \{x, x'\} \). Hence, we can separately decompose \( K_{2,n-2} \) and \( L_{n-2}^m \) into closed trails of length four by Theorem 4 and induction, respectively. Assume therefore that there does not exist a vertex of type one in \( G \), hence \( m \geq \frac{n}{2} \). Then, since \( m \) is an even number not greater than \( n - 2 \), either there exist \( x, y, z \in V(G) \) such that \( t(x) = 3 \) and \( t(y) = 2 = t(z) \) or \( t(u) \in \{2, 4\} \) for each \( u \in V(G) \). In the first case, we have \( G = (L_6^4, K_{6,n-6}).L_{n-6}^{m-4} \), where \( V(L_6^4) = \{x, x', y, y', z, z'\} \). Then we can decompose \( K_{6,n-6} \) into closed trails of length four by Theorem 4 and the remaining two graphs by induction. In the last case, when \( t(u) \in \{2, 4\} \) for each \( u \in V(G) \), we have \( m = \frac{n}{2} \) and \( \|G\| = \frac{n(n-1)}{2} \). On the other hand, since \( \tau = 4p \) is admissible for \( G \), \( \|G\| \equiv 0 \) (mod 4). Therefore \( n \equiv 0 \) (mod 8) and \( G = (L_8^4, K_{8,n-8}).L_{n-8}^{m-4} \), hence we can, as above, decompose these three graphs into closed trails of length four separately.
We assume from now on, that the sequence \( \tau \) is nonincreasing, i.e. \( a_1 \geq a_2 \geq \ldots \geq a_p \), and is not of the form \( 4^p \). Let also \( G = R_n.L_{n-4}^{m-2} \), where \( R_n \in \mathcal{R}_n, L_{n-4}^{m-2} \in \mathcal{L}_{n-4}^{m-2} \) and \( s = \|R_n\| \).

Case 1: For some \( i, s_i = s \). Then we can find a realization of \( \tau_1 = (a_1, \ldots, a_i) \) in \( R_n \) by Lemma 7 and then decompose \( L_{n-4}^{m-2} \) into closed trails of lengths \( a_{i+1}, \ldots, a_p \) by induction.

Case 2: For some \( i, s_{i-1} \leq s - 4 \) and \( s_i \geq s + 4 \). Let \( \tau_2 = (a_1, \ldots, a_{i-1}, a_i') \) and \( \tau_3 = (a_i', a_{i+1}, \ldots, a_p) \), where \( a_i' = s - s_{i-1} \geq 4 \) and \( a_i'' = a_i - a_i' \geq 4 \). Since \( a_i \geq 8 \) and \( \tau \) is nonincreasing, then \( a_j \geq 8 \) for all \( j < i \), hence we can find its realization in \( R_n = L_4^2.K_{4,n-4} \) by Lemma 7. Moreover, by the comment bellow the proof of this lemma, we can find this realization in such a way that a closed trail of length \( a_i' \) contains at least one vertex from the partition set of size \( n-4 \) of \( K_{4,n-4} \). Then, by induction, we find a realization of \( \tau_3 \) in \( L_{n-4}^{m-2} \) and permute its vertices in such a way that the trails of lengths \( a_i' \) and \( a_i'' \) meet at some vertex forming a trail of length \( a_i \).

Case 3: For some \( i, s_i = s + 2 \). If \( m \geq 4 \), we have \( G = G_1.L_{n-4}^{m-4} \), where \( G_1 = R_n.L_2 \). Then \( \|G_1\| = s + 2 \) and the sequence \( \tau_1 = (a_1, \ldots, a_i) \) is not of the form \( 4^p \) (because nonincreasing \( \tau \) is not). Hence we can find a realization of \( \tau_1 \) in \( G_1 \) by Lemma 8 and a realization of the remaining elements of \( \tau \) in \( L_{n-4}^{m-4} \) by induction. Assume therefore \( m = 2 \). Then \( G = G_2.L_2^{n-4} \), where \( G_2 = L_4^2.K_{4,n-4} \) and \( \|G_2\| = s_i - 4 \). If \( a_1 \geq 12 \), then we find realizations of sequences \( \tau_2 = (a_i - 8, a_{i+1}, \ldots, a_p) \) in \( K_{4,n-4} \) and \( L_{n-4}^2 \) by Theorem 4 and induction, respectively. Then it is enough to glue together the closed trail of length \( a_1 - 8 \) with \( L_2^0 \) and a closed trail of length four from \( L_{n-4}^2 \) to form the one of length \( a_1 \). Analogously, if there exist \( t_1, t_2 \leq i \) such that \( a_{t_1}, a_{t_2} \neq 6 \), then we find a \( K_{4,n-4} \)-realization of \( \tau_1 \) with elements \( a_{t_1}, a_{t_2} \) exchanged (or removed) by \( a_{i_1} - 4, a_{i_2} - 4 \) and an \( L_{n-4}^2 \)-realization of \( \tau_2 \). Now it is enough to glue together the closed trails of length \( a_{i_1} - 4, a_{i_2} - 4 \) with \( L_2^0 \) and a closed trail of length four from \( L_{n-4}^2 \), respectively. Consequently, we may assume \( \tau_1 = 6^i \) or \( \tau_1 = 8 \cdot 6^{i-1} \) or \( \tau_1 = 10 \cdot 6^{i-1} \) or \( \tau_1 = 6^{i-1} \cdot 4 \), hence \( a_2 = a_3 = 6 \). If \( t_p = 4 \), then we find a realization of \( \tau_4 = (a_1, a_p, a_4, \ldots, a_i) \) in \( K_{4,n-4} \) by Theorem 4 and we are done, since \( G = (L_4^2.K_{4,n-4}).L_2^0 \), where \( L_4^2 \) is a closed trail of length \( a_3 \) and we can find a realization of the remaining elements of \( \tau \) in \( L_2^0 \) by induction. Therefore, since \( \tau \) is nonincreasing, we may assume that almost all of its elements (except possibly one) are equal to six. In such a case we have \( G = (L_6^0.K_{6,n-6}).L_2^0 \), where \( L_6, K_{6,n-6} \) are decomposable into closed trails of length six and we can find an \( L_2^0 \)-realization of the remaining elements of \( \tau \) by induction.
Case 4: For some $i$, $s_i = s - 2$. In such a situation, if $a_p = 4$, we have $s_i + a_p = s + 2$ and we continue the proof the same way as in case 3. Therefore, we may assume $a_j ≥ 6$ for each $j$ and $s_i + a_p ≥ s + 4$. If additionally $a_1 > a_p$, we have $s_i - a_1 + a_p ≤ s - 4$ and we proceed the same way as in case 2. Hence, we are left with the case $τ = t'$ with $t ≥ 6$.

Case 4.1: $τ = 6p$. Then we first find by induction a realization of the sequences $6p^1 · 4$ in $L_{m-2}$, where a closed trail of length four we denote as $A_1 = (v_1, \ldots, v_4, v_1)$. Then we find a realization of the sequence $8 · 6p^2$ in $R_n = L_{4}^2 K_{4,n-4}$ by taking $L_{4}^2$ as one closed trail of length 6 and then decomposing $K_{4,n-4}$ into $p_2 - 1$ copies of $C_6$ and one $C_8$ by Theorem 9. We may assume $C_8 = (w_1, \ldots, w_8, w_1)$, where $w_1 = v_1$ and $w_5 = v_3$ (it is enough to permute the vertices of $K_{4,n-4}$). Then the union of such $A_1$ and $C_8$ can be easily decomposed into two cycles of length 6, namely of the form $(w_1, w_2, w_3, w_4, w_5, v_1, w_1)$ and $(w_1, v_2, w_5, w_6, w_7, w_8, w_1)$, see Figure 5. Since obviously $p = p_1 + p_2 + 2$, we receive a realization of $6p$ in $G$.

Case 4.2: $τ = t^p$ and $t ≥ 8$. As above, we first find a realization of the sequence $t^{p_1} · (t-2)$ in $L_{n-4}^m$, with $A_1 = (v_1, \ldots, v_{t-2}, v_1)$ being a closed trail of length $t - 2$. Then, by Lemma 7, we find a realization of the sequence $(t+2) · t^{p_2}$ in $R_n = L_{4}^2 K_{4,n-4}$ in such a way that a closed trail $A_2 = (w_1, \ldots, w_{t+2}, w_1)$ of length $t+2$ is a subset of $K_{4,n-4}$ (or is equal to the entire $R_n$). It is possible by the comment below the proof of that lemma. Therefore, since $t + 2 ≥ 10$, $A_2$ contains at least three vertices from the partition set of size $n-4$ of $K_{4,n-4}$, and we may assume $w_1, w_5$ are distinct such vertices. Moreover, since we can permute the vertices of $L_{n-4}^m$, we may assume $v_1 = w_1$ and $v_3 = w_5$. Then the union of such $A_1$ and $A_2$ can be, analogously as in the previous subcase, decomposed into two closed trails of length $t$, see Figure 5.

![Figure 5: Intersecting closed trails.](image)
4 Decomposition of $L_n$ with odd $n$

We can now prove a theorem which, together with Theorem 5, closes the subject of packing closed trails of even lengths into $M_n$ for an arbitrary $n$, see Theorem 14.

**Theorem 10** Graphs $L_n$ and $L'_n$ with odd $n$ are arbitrarily decomposable into closed trails of even lengths.

Let $n$ be an odd number and $G = L_n$ or $G = L'_n$. Let us fix a loopless vertex of $G$ in the case $n \equiv 1 \pmod{4}$ or any vertex with a loop otherwise and label it as $x$. Then take a subgraph of $G$ of the form $L^{m}_{n-1} \in L^m_{n-1}$, $m$ even, containing all the vertices of $G$ except $x$. Then $G = L^{m}_{n-1}G_x$, where $G_x$ has even size and is one of the form presented below, see Figure 6. Observe that $x$ is the only vertex that joins subsequent cells of the graph $G_x$, where by cells we mean triangles with possible loops at their bottom (as in Figure 6) and sometimes upper vertices. Since we expect to find a decomposition of $G_x$ into closed trails, we are particularly interested in subgraphs of $G_x$ consisting of sets of these cells glued at $x$. We introduce the following notion. Let $T_0$ stand for a loopless triangle, $T_1$ for a triangle with one loop at a bottom vertex (no matter which one), $T_2$ for a triangle with two loops at the bottom vertices and $T_{i+3}$, where $i \in \{0, 1, 2\}$, for one of the above $T_i$'s with a loop at its upper vertex added (e.g. $T_4$ will stand for a triangle with one loop at a bottom vertex and one at $x$), see Figure 7. Note that, by the choice of $x$, such a cell ($T_3$, $T_4$ or $T_5$) can only appear if $n \equiv 3 \pmod{4}$). Furthermore, we denote a subgraph of $G_x$ formed by subsequent cells $T_{i_1}, T_{i_2}, \ldots, T_{i_j}$ glued at $x$ by $T_{i_1}T_{i_2}\ldots T_{i_j}$. Such a subgraph is obviously a closed trail. We will also write $T^r_i$ for short instead of $T_{i_1}\ldots T_{i_j}$. For example, first five cells in Figure 6 (a) with a loop at $x$ added to the second cell, form a closed trail $T_1T_5T_0^3T_1$ of length 20. For simplicity, we take the order of the cells into account and usually decompose $G_x$ by cutting it into subsequent groups of cells.
We may however do everything the other way around and in our proof, we actually start by creating $G_x$ by gluing together closed trails of the described form (and of proper lengths) and this way receive $L_{n-1}^m$ by deleting all the edges of $G_x$ (together with the vertex $x$) from $G$. Clearly, we have $G = L_{n-1}^m G_x$ and we call such a graph $L_{n-1}^m$ a completion of $G_x$ in $G$. Analogously, we call $G_x$ a completion of $L_{n-1}^m$ in $G$. Notice that we have to be careful while creating $G_x$, since $m$ has to be even if we want to use Theorem 6 in our proof. Moreover, we cannot use too many loops to construct $G_x$ if $G = L_n'$.

To simplify the notation of the main proof, we formulate three simple observations. Let $G = L_n$ or $G = L_n'$ with odd $n \geq 7$ and let $\tau = (a_1, \ldots, a_p)$ be an even admissible sequence for $G$. Assume $G = L_{n-1}^m G_x$, where $L_{n-1}^m \in \mathcal{L}_{n-1}^m$, with even $m < n$ ($m < n - 2$ if $G = L_n'$), and $G_x$ are some subgraphs of $G$ created as described above. Then the following statements hold true.

**Observation 11** If there exists a permutation $\tau_1 = (b_1, \ldots, b_j, \ldots, b_p)$ of $\tau$ such that the sequence $\tau_1' = (b_1, \ldots, b_j)$ is realizable in $G_x$, then $\tau$ is realizable in $G$.

*Proof.* It is true by Theorem 6, since we can find a realization of the remaining elements of $\tau$ in $L_{n-1}^m$. ■

**Observation 12** If there exists a permutation $\tau_1 = (b_1, \ldots, b_j, \ldots, b_p)$ of $\tau$ such that a sequence $\tau_1'' = (b_1, \ldots, b_{j-1}, b'_j)$, where $b'_j = b_j - b_j' \geq 4$, is admissible and realizable in $G_x$ in such a way that $B_j'$ (the respective realization of $b_j'$) contains $T_2T_1T_0$ or $T_5T_1T_0$ as an induced subgraph, then $\tau$ is $G$-realizable.

*Proof.* By Theorem 6, we find a realization of $\tau_1'' = (b''_j, b_{j+1}, \ldots, b_p)$ in $L_{n-1}^m$ with $B''_j$ being the respective realization of $b''_j$. Since $B''_j$ contains all possible kinds of cells of $G_x$ (not taking a possible loop at $x$ into account), i.e. $T_0$, $T_1$ and $T_2$, then we can easily permute $n - 1$ vertices of $G_x$ (all except $x$) in such a way that $B'_j$ meets at least one vertex of $B''_j$. Then $B_j = B'_j B''_j$ is a closed trail of length $b_j$ and we receive a realization of $\tau$ in $G$. ■

**Observation 13** If $a_p > \frac{3}{2}n - \frac{5}{2}$, then $\tau$ is realizable in $G$.  

12
Proof. We have \( a_p \geq 3 \frac{n-1}{2} \). Assume first \( n \equiv 1 (mod \ 4) \). Then if \( a_p = 3 \frac{n-1}{2} \), we take such a subgraph \( L^m_{n-1} \in L^m_{n-1} \) of \( G \), that \( G_x \) being its completion in \( G \) is of the form \( T_0^r \). Then \( ||G_x|| = a_p \) and we are done by Observation 11. Analogously, if \( a_p = 3 \frac{n}{2} - 2 \), it is enough if \( G_x = T_2 T_0^r \). If finally \( a_p = a'_p + a''_p \), where \( a'_p = 3 \frac{n-1}{2} \) and \( a''_p \geq 4 \), we again take \( G_x = T_0^r \) as a realization of \( (a'_p) \). Then by Theorem 6 we find a realization of the remaining elements of \( \tau \) (and \( a''_p \)) in \( L^m_{n-1} \) and since \( G_x \) contains all the vertices of \( G \), we easily glue together the closed trails of lengths \( a'_p \) and \( a''_p \).

If now \( n \equiv 3 (mod \ 4) \), then \( a_p \geq 3 \frac{n-1}{2} + 1 \), since \( a_p \) is an even number. Notice that if we take such a subgraph \( L^m_{n-1} \in L^m_{n-1} \) of \( G \), that \( G_x \) being its completion in \( G \) is of the form \( T_3 T_0^r \), then \( ||G_x|| = 3 \frac{n-1}{2} + 1 \) and we prove that \( \tau \) is \( G \)-realizable the same way as in the previous paragraph. 

Proof of Theorem 10. We verified the cases for \( n \leq 11 \) using a computer program we created, hence we assume \( n \geq 13 \) is an odd number. Let \( G = L_n \) or \( G = L'_n \) and let \( \tau = (a_1, \ldots, a_p) \) be a nondecreasing even admissible sequence for \( G \), hence \( a_1 \leq a_2 \leq \ldots \leq a_p \). In the first part of the proof we show that if \( \tau \) consists of sufficiently many small numbers, it is easy to construct a proper \( G_x \) by gluing together short closed trails and then use Observation 11 to finish the proof (notice that in all the subsequent cases, whenever \( G = L'_n \), we use at most \( n - 3 \) loops, not taking the possible one at vertex \( x \) into account, to create \( G_x \)). In the second part we deal with the case when there exists a sufficiently big element in \( \tau \). Note first that for \( G = L^m_{n-1}, G_x \), a graph \( G_x \) consists of odd number of cells \( T_i \) (\( i \in \{0, \ldots, 5\} \)) if \( n \equiv 3 (mod \ 4) \) or of even number of \( T_i \)'s if \( n \equiv 1 (mod \ 4) \).

Case 1: \( a_i, \ldots, a_i = 6 \) and \( 6l > \frac{3}{2} n - \frac{9}{2} \). Then we take such a subgraph \( L^m_{n-1} \in L^m_{n-1} \) of \( G \), that its completion \( G_x \) is of the form \( T_3 T_0^r \) if \( n \equiv 3 (mod \ 4) \) or \( T_0^r \) otherwise. Notice that \( r \) and \( s \) are even numbers. Therefore, we can easily decompose \( G_x \) into closed trails of length 6, namely of the form \( T_0^2 \) (or \( T_3 \)) and since \( ||G_x|| \leq 3 \cdot \frac{n-1}{2} + 3 = \frac{3}{2} n + \frac{3}{2} \), there is enough \( a_i \)'s of length six to sum up to the size of \( G_x \), hence we are done by Observation 11.

Case 2: \( a_i, \ldots, a_i = 10 \) and \( 10l > \frac{5}{2} n - \frac{35}{2} \). Then we take \( G_x = T_3 T_0^2 T_2 \) if \( n \equiv 3 (mod \ 4) \) or \( G_x = T_1 T_0^2 T_1 T_0^2 T_2 \) otherwise, where \( G_x \) is a graph of order \( n \). Again \( r \) and \( s \) are even and we can easily decompose \( G_x \) into closed trails of length 10, namely of the form \( T_2^2, T_3 T_0^2 \) or \( T_1 T_0^2 \). Since \( ||G_x|| \leq \frac{5}{2} n - \frac{15}{2} \), we are, analogously as above, done by Observation 11.

Case 3: \( a_i, \ldots, a_i = 14 \) and \( 14l > \frac{7}{2} n - \frac{21}{4} \). We conduct the same reasoning
as in the two above cases to find \( G_x \) of order \( n \) being easily decomposable into closed trails of length 14. Since we need to use exactly \( \frac{n-1}{2} \) \( T' \)'s to construct \( G_x \), therefore, if \( \frac{n-1}{2} \equiv 0(\text{mod } 4) \) (hence \( n \equiv 1(\text{mod } 4) \)), then we construct \( G_x \) by gluing together trails \( T_2 T_3 \). If \( \frac{n-1}{2} \equiv 1(\text{mod } 4) \) (hence \( n \equiv 3(\text{mod } 4) \)), we take one trail \( T_3 T_2 \), two trails \( T_2 T_1 \) and trails \( T_2 T_3 \). If \( \frac{n-1}{2} \equiv 2(\text{mod } 4) \) (hence \( n \equiv 1(\text{mod } 4) \)), we take two trails \( T_2 T_1 \) and trails \( T_2 T_3 \). Finally, if \( \frac{n-1}{2} \equiv 3(\text{mod } 4) \) (hence \( n \equiv 3(\text{mod } 4) \)), we construct \( G_x \) from one trail \( T_3 T_2 \) and trails \( T_2 T_3 \). It is easy to check that in all the cases the completion of \( G_x \) in \( G \) is of the form \( L_{n-1}^m \in L_n \) for some even \( m \) and by our construction \(|G_x|\) \( \leq \frac{7}{4}n + \frac{35}{4} \), hence the assumption made in this case is sufficient to guarantee existing enough number of 14’s in \( \tau \) to add up to the size of \( G_x \). Therefore we are done by Observation 11.

**Case 4:** \( a_{ij} \equiv 0(\text{mod } 4) \) for \( i = 1, \ldots, l \) and \( a_{i1} + \ldots + a_{il} > 2n - 6 \). Then it is enough to take \( G_x = T_3 T'_1 \) if \( n \equiv 3(\text{mod } 4) \) or \( G_x = T'_1 \) otherwise. Since \( G_x \) consists of cells of size 4, we can easily cut it into closed trails of lengths divisible by 4 and by our assumption there is enough \( a_{ij} \equiv 0(\text{mod } 4) \) to cover \( G_x \). Therefore, if there exists \( k \leq l \) such that \( a_{i1} + \ldots + a_{ik} = 2n - 2 \) (= \(|G_x|\)), then we are done by Observation 11. If not, then there exists \( k < l \) and \( c, d > 0 \) such that \( a_{ik+1} = 4c + 4d \) and \( a_{i1} + \ldots + a_{ik} + 4c = 2n - 2 \). In this case, we find a realization of \( \tau'_1 = (a_{i1}, \ldots, a_{ik}, 4c) \) in \( G_x \). Moreover, similarly as in Observation 12, we can choose this realization in such a way that a closed trail of length \( 4c \) contains any of the cells of \( G_x \), i.e. either \( T_1 \) or \( T_3 \). In other words, this closed trail may contain any of the vertices of \( G \). Therefore, if using Theorem 6 we find a realization of the remaining elements of \( \tau \) (and \( 4d \)) in \( L_{n-1}^m \) being a completion of \( G_x \) in \( G \), then we can glue together the proper closed trails of lengths \( 4c \) and \( 4d \) forming a one of length \( a_{ik+1} \). This way we receive a realization of \( \tau \) in \( G \).

**Case 5:** None of the above occurs. Let \( \tau' = (b_1, \ldots, b_q) \) be a nondecreasing subsequence of \( \tau \) consisting of all its elements \( b_j \) such that \( b_j \equiv 2(\text{mod } 4) \) and \( b_j \geq 10 \). Since the inequality

\[
\frac{n(n+1)}{2} - 3 > (2n - 6) + \left( \frac{3}{2}n - \frac{9}{2} \right) + \left( \frac{5}{2}n - \frac{35}{2} \right) + \left( \frac{7}{4}n - \frac{21}{4} \right),
\]

where \(|G| \geq \frac{n(n+1)}{2} - 3\), holds for each \( n \), then, in view of the previous cases, at least one of the elements of the nondecreasing sequence \( \tau' \) is not smaller than 18, hence \( b_q \geq 18 \). Moreover, since the inequality

\[
\frac{n(n+1)}{2} - 3 > (2n - 6) + \left( \frac{3}{2}n - \frac{9}{2} \right) + \frac{5(n-1)}{2},
\]

14
holds for \( n \geq 9 \), then \( b_1 + \ldots + b_q > \frac{5(n-1)}{2} \).

Notice that since \( b_q \geq 18 \), then \( \tau \) is \( G \)-realizable by Observation 13 if \( n = 13 \). Therefore, we assume from now on, that \( n \geq 15 \) and \( b_q \leq \frac{5}{2}n - \frac{5}{2} \).

Now, we will again construct a proper subgraph \( G_x \) of \( G \) whose completion in \( G \) will be of the form \( L_n^m \in L_{n-1}^m \) for some even \( m \) and use Observation 11 or 12 to finish the proof. Let for each \( j < q \), \( l(j) \) denote the greatest number of cells \( T_i \) \( (i = 0, 1, 2) \) which we have to use to construct a closed trail of length \( b_j \), but under the condition that this trail consists of at least one loop. For instance, if \( b_j = 10 \), then we can construct a closed trail of this length in two ways, as \( T_2^2 \) and \( T_1 T_0^2 \), hence \( l(j) = 3 \), but we remember that there is also a closed trail of length 10 using one less triangle, namely \( l(j) - 1 = 2 \) triangles. Analogously, \( T_2^3 T_1 \) and \( T_2^2 T_0^2 \) are both of length 14, but \( l(j) = 4 \) for \( b_j = 14 \). For \( b_j = 18 \) we have three representations \( T_3^3 T_0, T_2 T_1 T_0^3, T_0^6 \) on different number of triangles, but only two first of them consist of at least one loop, hence \( l(j) = 5 \) in this case. However, whenever \( b_j \geq 22 \), we have at least three representations consisting of at least one loop, in particular on \( l(j), l(j) - 1 \) and \( l(j) - 2 \) triangles. For example \( l(j) = 7 \) for \( b_j = 22 \), because \( T_1 T_0^6 \) is of length 22, but \( T_2^3 T_0^2 \) and \( T_2^2 T_1 T_0 \) are also closed trails of length 22. We will use ”the widest” (consisting of maximal number of cells) of such closed trails to construct \( G_x \) so as to be able, in a way, squeeze them later if necessary. The requirement of one loop appearing in these closed trails is important in the case \( n \equiv 3 \) (mod 4), when there has to be a loop at vertex \( x \) in \( G_x \). Moreover, we define \( l(q) \) to be the number of \( T_i \)'s in a closed trail of length \( b_q \) of the form \( T_2 T_1 T_0 T_i T_0 T_i r_1 \), where \( i_1 \equiv b_q \) (mod 3), \( i_1 \in \{ 0, 1, 2 \} \) and \( r_1 \geq 1 \), hence \( l(q) = r_1 + 4 \). We use this representation because it is the widest one containing \( T_3 T_1 T_0 \) as an induced subgraph, see Observation 12. Note also that \( l(q) \geq 5 \) and, since \( b_q \geq b_j \) for all \( j < q \), then \( l(q) \geq l(j) - 1 \) for \( j < q \). Let now \( k < q \) be the smallest number for which

\[
\sum_{j=1}^{k} l(j) + l(q) \geq \frac{n-1}{2},
\]

where \( \frac{n-1}{2} \) is the number of triangles that we have to use to construct \( G_x \) (there exists such \( k \), since \( b_1 + \ldots + b_q > \frac{5(n-1)}{2} \)).

**Case 5.1:** \( l(1) + \ldots + l(k) = \frac{n-1}{2} \), (it is possible, since \( l(q) \) may be equal to \( l(k) \) - 1). Then we construct \( G_x \) by joining together closed trails of lengths \( b_j \), \( j = 1, \ldots, k \), on \( l(j) \) triangles described above (in the case when \( n \equiv 3 \) (mod 4), we use a slightly different closed trail of length \( b_1 \), namely we exchange either one \( T_1 \) to \( T_3 \) or one \( T_2 \) to \( T_4 \) in the described above representation). Since all \( b_j \) are even, it is also obvious that the completion of \( G_x \)
in $G$ will have an even number of loops, hence we are done by Observation 11.

Case 5.2: $l(1) + \ldots + l(k) + l(q) = \frac{n-1}{2}$. Then we are analogously done by Observation 11.

Case 5.3 $l(1) + \ldots + l(k) + l(q) = \frac{n-1}{2} + 1$. Then to construct $G_x$ we may use instead of the closed trail $T_2T_1T_0T_1T_0^{r_1}$ of length $b_q$, another one, namely $T_2^bT_1T_0^{r_1-1}$, also of length $b_q$, but consisting of one less triangle. Hence, we are again done by Observation 11.

Case 5.4: $l(1) + \ldots + l(k) + l(q) \geq \frac{n-1}{2} + 2$ and $l(1) + \ldots + l(k) \leq \frac{n-1}{2} - 3$. Let $R$ be such a number that $l(1) + \ldots + l(k) + R = \frac{n-1}{2}$, $R \geq 3$. If $R = 3$, then we take $b'_q = 12$, $b''_q = b_q - b'_q \geq 6$ and we construct $G_x$ by joining described closed trails of lengths $b_j$, $j = 1, \ldots, k$, on $l(j)$ triangles and a closed trail $T_2T_1T_0$ of length 12. This way a sequence $\tau'' = (b_1, \ldots, b_k, b'_q)$ is realizable in $G_x$ and we are done by Observation 12. If now $R = 4 + r_2$ for some $r_2 \geq 0$, then to construct $G_x$ we take the proper closed trails of lengths $b_j$, $j = 1, \ldots, k$, and a closed trail $T_2T_1T_0T_2T_0^{r_2}$, where $i_2 = i_1$ if $r_1 - r_2$ is an even number or $i_2 \in \{0, 1\}$ and $|i_2 - i_1| = 1$ otherwise. Observe that since $l(1) + \ldots + l(k) + l(q) \geq \frac{n-1}{2} + 2$, we have $r_1 - r_2 \geq 2$. Therefore, if we take $b''_q = ||T_2T_1T_0T_2T_0^{r_2}||$ and $b''_q = b_q - b'_q$, then $b''_q$ is an even number greater than four and a sequence $\tau'' = (b_1, \ldots, b_k, b'_q)$ is realizable in $G_x$, hence we are again done by Observation 12.

Case 5.5: $l(1) + \ldots + l(k) + l(q) \geq \frac{n-1}{2} + 2$ and $l(1) + \ldots + l(k) = \frac{n-1}{2} - 2$. Then instead of using a closed trail of length $b_1$ on $l(1)$ triangles to construct $G_x$, we use another one, on $l(1) - 1$ triangles, also described above. This way we have $(l(1) - 1) + l(2) + \ldots + l(k) = \frac{n-1}{2} - 3$ and (since $l(q) \geq 5$) $(l(1) - 1) + l(2) + \ldots + l(k) + l(q) \geq \frac{n-1}{2} + 2$, hence we get back to the case 5.4.

Case 5.6: $l(1) + \ldots + l(k) + l(q) \geq \frac{n-1}{2} + 2$ and $l(1) + \ldots + l(k) = \frac{n-1}{2} - 1$. If $k \geq 2$, then to construct $G_x$ we can use instead of closed trails of lengths $b_1, b_2$ on $l(1)$, $l(2)$ triangles, another ones, on $l(1) - 1$ and $l(2) - 1$ triangles. This way we have $(l(1) - 1) + l(2) - 1 + l(3) + \ldots + l(k) = \frac{n-1}{2} - 3$ and $(l(1) - 1) + l(2) - 1 + l(3) + \ldots + l(k) + l(q) \geq \frac{n-1}{2} + 2$, hence we get back to the case 5.4. If on the other hand $k = 1$ and $b_1 \geq 22$, we can use instead of a closed trail of length $b_1$ on $l(1)$ triangles, another one, on $l(1) - 2$ triangles to construct $G_x$. This way we have $(l(1) - 2) = \frac{n-1}{2} - 3$ and $(l(1) - 2) + l(q) \geq \frac{n-1}{2} + 2$, hence we again get back to the case 5.4. To finish the proof, notice that it is not possible that $k = 1$ and $b_1 \leq 18$, because then we would have $l(1) \leq 5$ and consequently $n \leq 13$. 

16
5 Final results and remarks

Here we sum up our results in two theorems second of which is a direct consequence of the first one.

**Theorem 14** Let \( \tau = (a_1, \ldots, a_p) \) be a sequence of positive even integers greater than two. Then we can edge-disjointly pack closed trails of lengths \( a_1, \ldots, a_p \) into \( M_r \), whenever \( \sum_{i=1}^{p} a_i \leq ||L_r|| \), except for the case \( \tau = (4, 4) \) and \( r = 4 \).

**Proof.** Let \( n = \sum_{i=1}^{p} a_i \). Since \( n \leq ||L_r|| \), then by Theorem 5 or 10 we may find a realization of \( \tau \) in \( L_r \) or \( L'_r \) if \( n \in \{||L'_r||, ||L_r||\} \) or a realization of \( \tau' = (a_1, \ldots, a_p, ||L_r|| - n) \) in \( L_r \) otherwise. This way, since \( L_r \) and \( L'_r \) are subgraphs of \( M_r \), we receive the desired packing. \( \square \)

**Theorem 15** Let \( G = C_{a_1} \cup \ldots \cup C_{a_p} \) be a simple 2-regular graph of order \( n = \sum_{i=1}^{p} a_i \), where \( a_1, \ldots, a_p \) are even numbers. Then \( c(G) = \lceil \sqrt{2n} \rceil - 1 \) if \( \frac{r^2}{2} < n \leq \left( \frac{r+1}{2} \right) \) for some odd \( r \) and \( c(G) = \lceil \sqrt{2n} \rceil \) in all other cases with one exception \( c(C_4 \cup C_4) = 5 \).

**Proof.** Since \( L_r \) is a maximal, in terms of size, Eulerian subgraph of \( M_r \) with even number of edges, it is obvious, we cannot pack closed trails of lengths \( a_1, \ldots, a_p \) in \( M_r \) if \( n > ||L_r|| \). Therefore, by Theorem 14, we have \( c(G) = r \) (\( c(C_4 \cup C_4) = 5 \)), where \( r \) is the smallest number such that \( n \leq ||L_r|| \). The solution of this problem yields the described result. \( \square \)

The problem remains still open if we admit odd cycles as components of 2-regular graphs. The main obstacle here is the fact that closed trails of length three cannot contain loops. On the other hand, we may assume, while looking for a realization of some \( \tau \) in \( L_n \), that in \( \tau \) there is enough \( a_i \)'s of proper lengths to cover all the loops of \( L_n \). Observe however, that even though the sequence \( \tau = (3, 3, 6, 6) \) comply with this condition for \( L_6 \) (each closed trail of length 6 may contain 3 loops), there exists no \( L_6 \)-realization of \( \tau \).

Another problem is the technique of decomposing graphs we used, basing on Theorem 4 for bipartite graphs, in which only closed trails of even lengths exist. However, using a slightly different approach to the subject of packing, together with some additional condition for closed trails of length three, should result in determining the irregular coloring number for all 2-regular graphs, but it is still to come.
References


