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Vertex-distinguishing edge-colorings of sums of paths

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Abstract

In the PhD thesis by Burris (Memphis (1993)), a conjecture was made concerning the number of colors $c(G)$ required to edge-color a simple graph $G$ so that each vertex has a distinct multiset of colors incident to it. We find the exact value of $c(G)$ - the irregular coloring number, and hence verify the conjecture when $G$ is a vertex-disjoint union of paths. We also investigate the point-distinguishing chromatic index, $\chi_0(G)$, where sets, instead of multisets, are required to be distinct, and determine its value for the same family of graphs.

1 Introduction

Consider a simple graph $G$. In [1] the following coloring problem was introduced. Let $C$ be a color set, and let $w: E(G) \rightarrow C$ be an edge-coloring of $G$. Let then $S(v)$ (or $S_w(v)$ if the coloring $w$ is not obvious) denote the multiset of colors assigned to the edges incident with $v \in V(G)$. We call this edge-coloring irregular or vertex-distinguishing if $S(u) \neq S(v)$ for any two distinct vertices $u, v \in V(G)$. It exists iff $G$ contains no more than one isolated vertex and no isolated edges. Such a graph we call vertex-distinguishing edge-colorable (vdec-graph), while the minimal number of colors necessary to obtain its irregular edge-coloring we call an irregular coloring number and denote by $c(G)$.

Let $n_d$ (or $n_d(G)$) denote the number of vertices of degree $d$ in $G$. Note that if there is an irregular edge-coloring of $G$ with $k$ colors, then, by the
standard combinatorial formula for the number of multisets of a given size, we must have that
\[
\binom{k + d - 1}{d} \geq n_d
\]
for each \(d \geq 1\). The following conjecture was posed by Burris in [4].

**Conjecture 1** Let \(G\) be a vdec-graph and let \(k\) be the minimum integer such that \(\binom{k + d - 1}{d} \geq n_d(G)\) for \(1 \leq k \leq \Delta(G)\). Then \(c(G) = k\) or \(k + 1\).

In the general case this conjecture appears to be difficult. Several results, mostly for connected graphs, are however included in [4]. In [6] we investigated some aspects of the irregular edge-coloring of 2-regular graphs, which are simply (vertex-disjoint) sums of cycles. In this paper, we prove that Conjecture 1 holds for the sums of paths. In our research we apply a similar method as the one described in [3], where the graph parameter \(\chi'_s(G)\), in the case when \(\Delta(G) = 2\), is investigated. It is called the *strong coloring number* and may be viewed as the modification of \(c(G)\) by the restriction that the edge-coloring has to be proper (see also [4, 5]).

It will appear in the next section that in the case when \(G\) is a disjoint union of paths, the problem of irregular edge-coloring is equivalent to a certain problem of packing of the line graph of \(G\) into a special pseudograph. We solve it in Section 3. In the last section we study the *point-distinguishing chromatic index*, \(\chi_0(G)\) (see [7]), which is another modification of \(c(G)\) where sets, instead of multisets, are required to be distinct. We determine its value also for the sums of paths.

## 2 Paths packing problem

Let \(M_n\) denote the complete graph \(K_n\) with a single loop at each vertex added. Though \(M_n\) is actually a pseudograph, we shall call it a graph. Moreover, write \(P_{n+1}\) for a path of length \(n\) (on \(n + 1\) vertices) and \(P(v_1, v_2, \ldots, v_{n+1})\) for the trail of length \(n\) on the vertices \(v_i\) and with edges \(v_iv_{i+1}\). We do not require the \(v_i\) to be distinct. We call the trail *open* if \(v_1 \neq v_{n+1}\) and we call it *closed* otherwise. For any two graphs \(G_1\) and \(G_2\), write \(G_1 \cup G_2\) for the vertex-disjoint union of \(G_1\) and \(G_2\).

If \(G_1\) and \(G_2\) are graphs, a *packing* of \(G_1\) into \(G_2\) is a map \(f : V(G_1) \to V(G_2)\) such that \(xy \in E(G_1)\) implies \(f(x)f(y) \in E(G_2)\) and the induced map on edges \(xy \mapsto f(x)f(y)\) is an injection from \(E(G_1)\) to \(E(G_2)\). We do not require \(f\) to be injective on vertices, so if \(G_1\) contains a path, its image in \(G_2\) will be a trail.
In this paper we investigate the case when a graph $G$ is a (vertex-disjoint) union of $p$ paths, hence $G = P_{l_1 + 2} \cup \ldots \cup P_{l_p + 2}$ with $l_i \geq 1$ ($G$ is a vdec-graph and we ignore the presence or absence of an isolated vertex, since this does not influence the coloring). The line graph $L(G) = P_{l_1 + 1} \cup \ldots \cup P_{l_p + 1}$ of such a graph is also a union of $p$ paths. If $G$ is given a vertex-distinguishing edge-coloring by $n$ colors, then we get a packing of $L(G)$ as $p$ edge-disjoint trails in $M_n$. Each edge of $G$ corresponds to a vertex of $L(G)$ which is mapped to a (color) vertex of $M_n$. Since the coloring is vertex-distinguishing, the trails being images of the paths are indeed edge disjoint in $M_n$. Moreover they are open, since the $2p$ endpoints of the paths have to be mapped to distinct vertices. Conversely, if we have a packing of $L(G)$ into $M_n$ such that each endpoint of the paths in $L(G)$ is mapped to a different vertex, then we can color each edge of $G$ with the image of the corresponding vertex of $L(G)$ in $M_n$. The obtained edge-coloring is vertex-distinguishing, hence the value of $c(G)$ is equal to the smallest $n$ such that a packing of $L(G)$ into $M_n$ with all endpoints mapped to distinct vertices exists.

3 Irregular edge-coloring

We make use of the following result by Balister from [2] to solve the packing problem to which our problem was reduced and thus determine the exact value of $c(G)$ in the described case.

**Theorem 2** Let $L = \sum_{i=1}^{p} t_i$, $t_i \geq 3$, with $L = \binom{n}{2}$ when $n$ is odd and $(\frac{n}{2}) - \frac{n}{2} - 2 \leq L \leq (\frac{n}{2}) - \frac{n}{2}$ when $n$ is even. Then we can write some subgraph of $K_n$ as an edge-disjoint union of closed trails of lengths $t_1, \ldots, t_p$.

**Theorem 3** The following conditions are both necessary and sufficient for packing $\bigcup_{i=1}^{p} P_{l_i + 1}$, $l_i \geq 1$, into $M_n$ with endpoints mapped to distinct vertices:

(1) $L \leq \binom{n+1}{2} - \frac{r}{2}$ if $r$ (or $n$) is even,

(2) $L \leq \binom{n+1}{2} - p$ if $r$ (or $n$) is odd,

where $n = 2p + r$, $r \geq 0$, and $L = \sum_{i=1}^{p} l_i$. In particular $L \leq \binom{n}{2}$ is always sufficient.

**Proof.** We verified the cases for $n \leq 9$ by a computer program we created\(^1\) (we might have done it without using a computer, but then the proof gets

\(^1\)The source code along with other necessary files are available at http://home.agh.edu.pl/~cichacz/ang/preprints.php.
longer and more unclear), thus let \( n \geq 10 \).

First we prove that the conditions are necessary.

Clearly, we cannot pack paths of total length \( L \) greater than the size of \( M_n \), \( \binom{n+1}{2} \). Moreover, if \( G' \) is the image of the described packing in \( M_n \), it consists of \( p \) open trails, whose ends form a set of \( 2p \) vertices of odd degrees in \( G' \). The remaining \( r \) vertices have even degrees in \( G' \). Therefore, if \( r \) is odd (hence \( n \) is odd and the degrees of all vertices of \( M_n \) are even), we must delete at least \( p \) edges from \( M_n \) to obtain \( G' \). Analogously, if \( r \) is even (hence \( n \) is also even and the degrees of all vertices of \( M_n \) are odd), we need to remove at least \( \frac{n}{2} \) edges from \( M_n \) to obtain \( G' \).

Now we prove the sufficiency of the conditions by induction on \( n \).

Let \( l_1 \geq l_2 \geq \ldots \geq l_p \). If all the paths are of length one, we are done, since in \( M_n \) there is a set of \( \lceil \frac{n}{2} \rceil \geq p \) independent edges. Therefore, we may assume \( l_1 \geq 2 \).

Let us first consider the case \( p = 1 \). If \( n \) is odd (even) and \( l_1 = \binom{n+1}{2} - 1 \) (\( l_1 = \binom{n+1}{2} - \frac{n}{2} + 1 \)), remove one edge from the Eulerian graph \( M_n \) (remove \( \frac{n}{2} - 1 \) independent edges from \( M_n \)) to form the desired open trail. If \( l_1 = \binom{n+1}{2} - q \) \( l_1 = \binom{n+1}{2} - \frac{n}{2} + q \), where \( 1 \leq q \leq n \), it is enough to remove \( q \) loops from the trail described above. Finally, if \( l_1 \leq \binom{n}{2} - 2 \) \( l_1 \leq \binom{n}{2} - \frac{n}{2} \) and \( l_1 \geq 4 \), we first find a closed trail \( T \) of length \( l_1 - 1 \) in \( K_n \) by Theorem 2. Clearly, there must be some edge \( uv \in E(K_n) \) that does not belong to this trail with \( u \) being one of its vertices. Adding this edge to \( T \) yields an open trail of length \( l_1 \). If \( l_1 \leq 3 \), the result is obvious.

Let then \( p \geq 2 \) and \( l_0 = l_1 + l_2 - 2 \). Since \( l_1 \geq 2 \), we have \( l_0 \geq 1 \). Assume first that the sum of the lengths of the paths is relatively small, so that the \( p - 1 \) paths of lengths \( l_0, l_3, l_4, \ldots, l_p \) satisfy the assumptions of the theorem for \( M_{n-2} \). Then we pack \( P_{l_0+1} \cup \bigcup_{i=3}^{l_p} P_{l_i+1} \) into \( M_{n-2} \) (\( M_n \) with two vertices, say \( a \) and \( b \), removed) by induction. Let \( u \) and \( v \) be the ends of the trail of length \( l_0 \) in \( M_{n-2} \) and let \( u' \) be a vertex of this trail such that the distance along the trail between \( u \) and \( u' \) equals \( l_1 - 1 \) (hence the distance along the trail between \( v \) and \( u' \) equals \( l_2 - 1 \)). Then, by adding the edge \( u'a \) to the part of this trail between \( u \) and \( u' \), we obtain a trail of length \( l_1 \) with endpoints \( u, a \), and by adding the edge \( bu' \) to the rest of the trail of length \( l_0 \), we obtain a trail of length \( l_2 \) with endpoints \( v, b \). Thus we obtain the desired packing.

Assume therefore that the sum of the lengths of the paths is relatively large, so that the lengths \( l_0, l_3, l_4, \ldots, l_p \) do not satisfy the assumptions of the theorem for \( M_{n-2} \) \( (n \geq 10) \). Then however, it is easy to check that the length of at least one path, \( l_1, l_2, \ldots, l_{p-1} \) or \( l_p \), must exceed 7, hence \( l_1 \geq 8 \).
and $l_0 - 6 \geq 1$. Let us reduce $l_0$ by 6 if $n$ is odd or reduce it by 5 if $n$ is even. Then keep reducing $l_i$, $i \geq 3$, and what is left of $l_0$ by multiples of four until we have reduced all the lengths to at most four or until we have removed a total (taking into account the initial reduction by 6 or 5) length of $2n - 4$ if $n$ is odd, or $2n - 3$ if $n$ is even. (Actually, since the sum $l_0 + \sum_{i=3}^{p} l_i$ is relatively large and $n \geq 10$, this process of reducing stopes always in the second case.) Hence, if we denote the reduced lengths by $l'_0, l'_3, \ldots, l'_p$, we have $l'_i \geq 1$ for all $i$ and $l'_0 + \sum_{i=3}^{p} l'_i \leq \max\{4(p - 1), L - (2n - 2)\}$ if $n$ is odd, or $l'_0 + \sum_{i=3}^{p} l'_i \leq \max\{4(p - 1), L - (2n - 1)\}$ if $n$ is even. Therefore, the paths $P_{l_0 + 1}, P_{l_3 + 1}, \ldots, P_{l_p + 1}$ satisfy the assumptions of the theorem for $M_{n-2}$ (since we have now $p - 1$ paths, $r$ remains the same) and we may pack them into this graph by induction. Assume $M_{n-2}$ was formed of $M_n$ by removing vertices $a$ and $b$ from it and let the trail of length $l'_0$ has endvertices $u$ and $v$ in $M_{n-2}$. Fix a vertex $u'$ on this trail which is at distance at most $l_1 - 6$ along the trail from $u$ and distance at most $l_2 - 1$ from $v$ and so that this distance from $v$ is equivalent to $(l_2 - 1) \mod 2$. It is possible since $l'_0 \leq (l_1 - 6) + (l_2 - 1)$ and $l'_0 \geq 1$. Now, we will use all (except $ab$ if $n$ is odd) the edges (and loops) incident with $a$ or $b$ to complete the trails, so that they make up the desired packing.

First, for each $P_{l_i + 1}$ with $i \geq 3$ that have been shortened fix an endpoint vertex $v_i$ of the trail of length $l'_i$ in $M_{n-2}$ not equal to $u'$. Now pick $\frac{l_i - l'_i - 2}{2}$ paths of length two of the form $P(a, x, b)$ where $x$ is neither $u'$ nor any of $v_i$. Joining these paths with the edges $v_i a, v_i b$ and the trail of length $l'_i$ gives a trail of length $l_i$ in $M_n$ with the same endvertices as the trail of length $l'_i$ in $M_{n-2}$. Now we need only to form two trails corresponding to $P_{l_1 + 1}$ and $P_{l_3 + 1}$ in $M_n$. A trail of length $l_2$ we construct by joining the part of the trail of length $l'_0$ between $v$ and $u'$ with the edge $u'a$ and with some number of the paths of length two between $a$ and $b$. This trail has endvertices $v$ and $a$ or $b$. Similarly, a trail of length $l_1$ we construct by joining subsequently the part of the trail of length $l'_0$ between $u$ and $u'$ with the edge $u'b$, then with the loop at $b$, one of the remaining paths of length two between $a$ and $b$, the loop at $a$, and with some (all, actually) of the rest of the paths of length two between $a$ and $b$. If $n$ is even, we additionally add the edge $ab$ to the trail at the end. By our construction, it has endvertices $u$ and $a$ or $b$ distinct (since $l_2 + l_1 = l'_0 + 1 + 7 + 4t$ if $n$ is odd or $l_2 + l_1 = l'_0 + 1 + 6 + 4t$ otherwise, $t \geq 0$) from the endvertices of the trail of length $l_2$. \[\square\]

Note that if $G = \bigcup_{i=1}^{p} P_{l_i + 2}$ with $L = \sum_{i=1}^{p} l_i$, the inequality (1) for $d = 1$ is of the form $k \geq 2p$, hence provides us with the restriction that we need to use at least as many colors as there are the endpoints of the paths. Analogously,
since in $G$ there are exactly $L$ vertices of degree 2, we obtain $\binom{k+1}{2} \geq L$ for $d = 2$. Consequently, Conjecture 1 is valid for sums of paths.

**Corollary 4** Let $G$ be a vertex-disjoint union of paths of lengths at least two. Let $n_1(G) \leq k$ and $n_2(G) \leq \binom{k+1}{2}$ with $k$ chosen as small as possible. Then $c(G) = k$ or $k + 1$. □

### 4 Point-distinguishing coloring

The concept of the point-distinguishing coloring, an edge-coloring distinguishing all vertices by sets of adjacent colors, was introduced in [7] by Harary and Plantholt. Note that since a vertex of degree $d$ in a graph $G$ may obtain a set of colors consisting of at most $d$ elements, we must have that

$$\binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{d} \geq n_1(G) + n_2(G) + \ldots + n_d(G)$$

for $1 \leq d \leq \Delta(G)$ if there is a point-distinguishing coloring of $G$ by $k$ colors. Several results concerning some classes of connected graphs can be found in [7]. We focus on the case when a vdec-graph $G$ does not have to be connected. Observe that

$$c(G) \leq \chi_0(G) \leq \chi'_s(G),$$

where $c(G)$ and $\chi_0(G)$ coincide when 2-regular graphs are considered, see [6]. Assume then $G = P_{l_1+2} \cup \ldots \cup P_{l_p+2}$, with $L = \sum_{i=1}^{p} l_i$, is a sum of $p$ paths of lengths at least two. Taking $d = 1$ and $d = 2$ in the inequality (2), we obtain $2p \leq k$ and $|G| = 2p + L \leq \binom{k+1}{2}$. Note that the problem of the point-distinguishing coloring of $G$ is equivalent to almost the same packing problem as in the case of irregular edge-coloring, with one new restriction. Namely, we additionally require for each endpoint of a path from $L(G)$, that if it is mapped to a vertex $v \in V(M_n)$, then the loop at $v$ does not appear in the image of the packing.

**Theorem 5** The following conditions are both necessary and sufficient for packing $\bigcup_{i=1}^{p} P_{l_i+1}$, $l_i \geq 1$, into $M_n$ with endpoints mapped to distinct vertices and with loops at these vertices not appearing in the image of the packing:

1. $L = \binom{n}{2}$ or $L \leq \binom{n}{2} - 3$ if $r = 0$,
2. $L \leq \binom{n+1}{2} - \frac{r}{2} - 2p$ if $r$ (or $n$) is even ($r > 0$),
3. $L \leq \binom{n+1}{2} - p - 2p$ if $r$ (or $n$) is odd,

where $n = 2p + r$, $r \geq 0$, and $L = \sum_{i=1}^{p} l_i$. In particular $L \leq \binom{n}{2} - 2p$ is always sufficient.
**Proof.** It is enough to make some modifications in the proof of Theorem 3. Here we only outline these changes. The cases for $n \leq 9$ were verified by a computer program we created\(^2\), thus let $n \geq 10$.

The necessity of the conditions follows by almost the same argument as in the mentioned proof. The subtraction of additional $2p$ in the above inequalities (2) and (3) corresponds to $2p$ loops that cannot appear in the image of the packing, while for $r = 0$ (hence $n = 2p$ and $\binom{n}{2} = \binom{n+1}{2} - 2p$), we need to remove at least three or neither of the edges from $M_n$ to obtain an even number of vertices of odd degree in the resulting image of the packing.

Now we comment on the proof of the sufficiency of the conditions. For $p = 1$ we only remove the two loops at the ends of the trail of length $l_1 = \binom{n+1}{2} - 1$ ($l_1 = \binom{n+1}{2} - \frac{n}{2} + 1$ if $n$ is even) described at the beginning of the paragraph. The rest remains (almost) the same. Let then $p \geq 2$. If the paths of lengths $l_0, l_3, l_4, \ldots, l_p$ satisfy the assumptions of this theorem for $M_{n-2}$, the proof does not change. Assume then this is not the case. The main difference in the rest of the proof rely on the change that we initially reduce $l_0$ by 4 (instead of 6) if $n$ is odd, or by 3 (instead of 5) if $n$ is even (it is enough that $l_1 \geq 6$). Analogously, we reduce the lengths of the paths by at most $2n - 6$ (not $2n - 4$) if $n$ is odd, or $2n - 5$ (not $2n - 3$) if $n$ is even.

Then, while fixing $u'$ on the trail of length $l'_0$, we require it to be at distance at most $l_1 - 4$ (instead of $l_1 - 6$) along the trail from $u$, and finally, at the end of the proof, we omit the loops at $a$ and $b$ while constructing a trail of length $l_1$.

**Corollary 6** Let $G$ be a vertex-disjoint union of paths of lengths at least two. Let $n_1(G) \leq k$ and $n_1(G) + n_2(G) \leq k + \binom{k}{2}$ with $k$ chosen as small as possible. Then $\chi_0(G) = k$ or $k + 1$.

**References**


\(^2\)The source code along with other necessary files are available at http://home.agh.edu.pl/~cichacz/ang/preprints.php.


