RLC Transient Response

1. Introduction

The student will analyze an RLC circuit. A unit step input will excite this circuit, producing a transient voltage response across all circuit elements. These responses will be analyzed by theory, simulation and experimental results. The primary response properties of concern are initial value and final voltage values along with voltage measurements at intermediate steps.

Equations that govern selected component transient responses will be computed piecewise. To construct these equations the student will need to determine the damping coefficient, natural dampening frequency, initial capacitor voltage and final capacitor voltage. Knowing these fundamental RLC circuit properties, the student will be able to determine the roots of the circuit's characteristic equation, basic circuit response, and corresponding transient equation. Once all these pieces are brought together to form a transient response equation, the student will compute the expected voltage at periodic time intervals. The circuit will be simulated by the student and then constructed, whereas oscilloscope measurements will be taken at intervals corresponding to those used in theoretical calculations.

2. Background

A series RLC circuit may be modeled as a second order differential equation. Finding the solution to this second order equation involves finding the roots of its characteristic equation. The characteristic equation modeling a series RLC is $s^2 + s\left(\frac{R}{L}\right) + \frac{1}{LC} = 0$.

This equation may be written as $s^2 + 2\alpha s + \omega_0{}^2 = 0$ with the following representations:

Damping Factor = $\alpha = \frac{R}{2L}$ Undamped Natural Frequency = $\omega_0 = \frac{1}{\sqrt{LC}}$

Characteristic Equation Roots = $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}$

Knowing the above RLC circuit properties along with initial and final transient values for voltage or current then enables the transient capacitor voltage or transient inductor current to be calculated. This is done using the following equations (where infinity in this case will refer to the value when the circuit reaches steady-state before changing again):

2.1. Transient Capacitor Voltage for a Unit Step Input to a Series RLC Circuit

Overdamped: $\alpha > \omega_0$, $v_C(t) = v_C(\infty) + A_1 e^{S_1 t} + A_2 e^{S_2 t}$ V

Critically Damped: $\alpha = \omega_0$, $v_C(t) = v_C(\infty) + (A_1 + A_2 t)e^{-\alpha t}$ V

Underdamped: $\alpha < \omega_0$, $v_C(t) = v_C(\infty) + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$ V *

*Note the difference between ω_o and ω_d in the equations.

2.2. Transient Inductor Current for a Unit Step Input to a Series RLC Circuit

Overdamped: $\alpha > \omega_0$, $i_L(t) = i_L(\infty) + B_1 e^{S_1 t} + B_2 e^{S_2 t}$ A

Critically Damped: $\alpha = \omega_0$, $i_L(t) = i_L(\infty) + (B_1 + B_2 t)e^{-\alpha t}$ A

Underdamped: $\alpha < \omega_0$, $i_L(t) = i_L(\infty) + (B_1 \cos \omega_d t + B_2 \sin \omega_d t)e^{-\alpha t}$ A

where the damped natural frequency is expressed as $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$. In order to calculate the initial conditions, the equation relating the voltage and current through an inductor will also be needed. Since the two elements are in series, this can be rewritten as: $\frac{d[V_C(0)]}{dt} = \frac{I_L(0+)}{C}$

3. Prelab

The prelab is broken down into a set of steps. For each step, you should **record your answer in the appropriate table in the Data Entry** section at the end of this document. Also, *obtain your lab instructor's signature* before proceeding to Section 4.

3.1. Overdamped voltage transient response of capacitor in RLC circuit

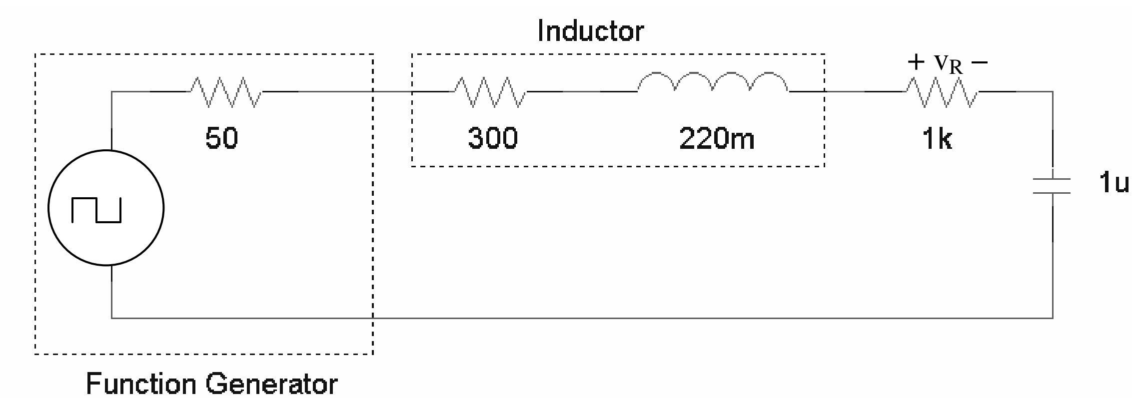

Figure 1: RLC Circuit with 1kΩ Resistor

A. Suppose the RLC circuit in Figure 1 has component values as displayed in the figure. Assume the function generator produces a square wave with a peak-to-peak amplitude of -5 to + 5 volts, and a frequency of 50 Hz. Compute the damping factor, α, and the undamped natural frequency, ω_0. Determine the initial voltage across the capacitor, $v_C(0^+)$, and the final voltage across the capacitor, $v_C(\infty)$. Compute the characteristic roots, $s_{1,2}$, utilizing α and ω_0, and state the type of damping.

B. The damping factor α should be greater than the undamped natural frequency ω_0, thus the circuit is overdamped and the capacitor's voltage transient can be expressed by the following equation with the two unknowns A_1 & A_2:

$v_C(t) = v_C(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$ Volts

Evaluating at time 0, the following relationship is formed:

$v_C(0) =$ ______ $= v_C(\infty) + A_1 e^0 + A_2 e^0$

Display all intermediate steps leading to a solution for $A_1 + A_2$.

C. Another equation is required to solve for both unknowns, thus the derivative of the capacitor voltage transient equation is taken.

$$\frac{d[V_C(t)]}{dt} = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$$

Again, evaluating at time 0, the following relationship is formed:

$$\frac{d[V_C(0^+)]}{dt} = \frac{i_L(0+)}{C} = \_\_\_\_\_ = s_1 A_1 e^0 + s_2 A_2 e^0$$

Display all intermediate steps leading to a solution for A_1 & A_2.

D. Having solved for the two unknown variables, write the full equation for $v_C(t)$. Evaluate $v_C(0.5\text{ mS})$, $v_C(1\text{ mS})$ and $v_C(2\text{ mS})$

3.2. Underdamped voltage transient response of capacitor in RLC circuit

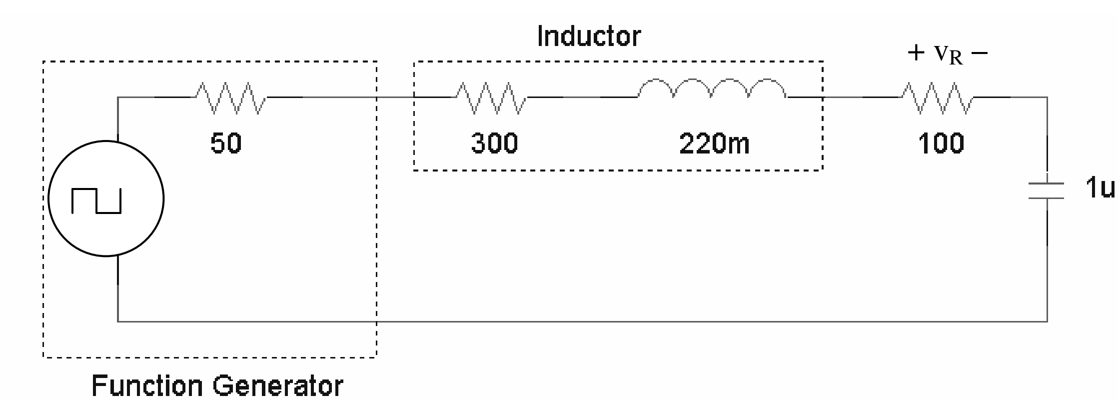

Figure 2: RLC Circuit with 100Ω Resistor

E. Compute the damping factor, α, and the undamped natural frequency, ω_0, for the circuit in Figure 2. Determine the initial voltage across the capacitor $v_C(0^+)$ and the final voltage across the capacitor $v_C(\infty)$. Compute the characteristic roots, $s_{1,2}$, utilizing α and ω_0, and state the type of damping.

F. The damping factor, α, should be less than the undamped natural frequency, ω_0. Thus, the circuit is underdamped and the capacitor's transient voltage can be expressed by the following equation with the two unknowns, B_1 & B_2.

$v_C(t) = v_C(\infty) + (B_1 \cos \omega_d t + B_2 \sin \omega_d t)e^{-\alpha t}$ Volts

Evaluating at time 0, the following relationship is formed:

$v_C(0) =$ _____ $= v_C(\infty) + (B_1 \cos(0) + B_2 \sin(0))\, e^0$ Volts

Display all intermediate steps leading to a solution for B_1.

G. Another equation is required to solve for B_2. Thus, the derivative of the capacitor voltage transient equation is taken.

$$\frac{d[V_C(t)]}{dt} = (B_1 \cos \omega_d t + B_2 \sin \omega_d t)(-\alpha e^{-\alpha t}) + (-B_1 \omega_d \sin \omega_d t + B_2 \omega_d \cos \omega_d t)(e^{-\alpha t})$$

Again, evaluating at time 0, the following relationship is formed:

$$\frac{d[V_C(0^+)]}{dt} = \frac{i_L(0^+)}{C} = \_\_\_\_ = (B_1 \cos \omega_d t + B_2 \sin \omega_d t)(-\alpha e^{-\alpha t}) + (-B_1 \omega_d \sin \omega_d t + B_2 \omega_d \cos \omega_d t)(e^{-\alpha t})$$

Display all intermediate steps leading to a solution for B_1 & B_2.

H. Having solved for the two unknown variables, write the full equation for $v_C(t)$. Evaluate $v_C(0.5 \text{ mS})$, $v_C(1 \text{ mS})$ and $v_C(2 \text{ mS})$.

3.3. Underdamped voltage transient response of resistor in RLC circuit

Suppose we want to find the voltage transient response of the 100 Ω resistor in Figure 2. The simplest way to solve for the resistor voltage transient is to find the transient circuit current and multiply by the resistor's resistance. Since we are analyzing a series circuit, we will find the transient inductor current and multiply by 100 Ω.

I. Compute the damping factor, α, and the undamped natural frequency, ω_0. Determine the initial current through the resistor, $i_R(0^+)$ (= $i_L(0^+)$), and then convert this value to the initial voltage, $v_R(0^+)$, using Ohm's Law. Similarly, the final voltage across the resistor in steady-state, $i_R(\infty)$ (= $i_L(\infty)$). Compute the characteristic roots, $s_{1,2}$, utilizing α and ω_0, and state the type of damping.

J. The damping factor, α, should be less than the undamped natural frequency, ω_0. Thus, the circuit is underdamped and the inductor's transient current can be expressed by the following equation with the two unknowns C_1 & C_2:

$i_L(t) = i_L(\infty) + (C_1 \cos \omega_d t + C_2 \sin \omega_2 t)e^{-\alpha t}$ Amps

Evaluating at time, 0 the following relationship is formed:

$i_L(0) = \_\_\_\_\_ = i_L(\infty) + (C_1 \cos 0 + C_2 \sin 0)e^0$ Amps

Display all intermediate steps leading to a solution for C_1.

K. Another equation is required to solve for C_2. Thus, the derivative of the inductor current transient equation is taken:

$$\frac{d[i_L(t)]}{dt} = (C_1 \cos \omega_d t + C_2 \sin \omega_2 t)(-\alpha * e^{-\alpha t}) + (-C_1 \omega_d \sin \omega_d t + C_2 \omega_d \cos \omega_d t)(e^{-\alpha t})$$

Evaluating at time 0, the following relationship is formed:

$$\frac{d[i_L(t)]}{dt} = \frac{v_L(0^+)}{L} = \_\_\_\_\_ = (C_1 \cos 0 + C_2 \sin 0)(-\alpha * e^0) + (-C_1 \omega_d \sin 0 + C_2 \omega_d \cos 0)(e^0)$$

Display all intermediate steps leading to a solution for C_1 & C_2.

L. Having solved for the two unknown variables, write the full equation for $v_R(t)$. Remember that you need to multiply $v_R(t)$ by the 100 Ω. Evaluate $v_R(0.5\text{ mS})$, $v_R(1\text{ mS})$, and $v_R(2\text{ mS})$.

4. Experimental Procedure

During the experimental procedure, make sure to **fill in the appropriate portions of the Tables 1-3 in the Data Entry** section and *obtain your lab instructor's signature* upon completion of the section (showing screen shots to him/her where appropriate).

4.1. Equipment

- Tektronix TDS 3012B Digital Phosphor Oscilloscope
- Agilent 33120A Waveform Generator
- Resistors, Inductors, Capacitors as Needed

4.2. Overdamped voltage transient response of capacitor in RLC circuit

Construct the circuit in Figure 1. Set the function generator to produce a square wave with a peak-to-peak amplitude of -5V to + 5V. Select the period large enough for the circuit to reach steady-state (e.g. f ≈ 40 -60 Hz). Using the horizontal bars, measure the initial capacitor voltage, $v_C(0^+)$, the final capacitor voltage, $v_C(\infty)$, and the capacitor voltage at 0.5 mS, 1 mS and 2 mS. Also, describe the type of damping you observe.

Include a screenshot of the circuit response in your lab write-up.

4.3. Underdamped voltage transient response of capacitor in RLC circuit

Construct the circuit in Figure 2. The function generator should again model a step of sufficient period as in Subsection 4.2. Using the horizontal bars, measure the initial capacitor voltage $v_C(0^+)$, the final capacitor voltage, $v_C(\infty)$, and the capacitor voltage at 0.5 mS, 1 mS and 2 mS. Also, describe the type of damping you observe.

Include a screenshot of the circuit response in your lab write-up.

4.4. Underdamped voltage transient response of resistor in RLC circuit

Using the horizontal bars, measure the initial resistor voltage, $v_R(0^+)$, the final resistor voltage, $v_R(\infty)$, and the resistor voltage at 0.5 mS, 1 mS and 2 mS. Also, describe the type of damping you observe.

Include a screenshot of the circuit response in your lab write-up.

5. Simulated Procedure

Generate a computer simulation modeling the capacitor voltage transient of the RLC circuit for Figures 1 and 2. Measure the initial capacitor voltage, $v_C(0^+)$, the final capacitor voltage, $v_C(\infty)$, and the capacitor voltage at 0.5 mS, 1 mS, and 2 mS. **Record your values in the appropriate sections of Tables 1 and 3**.

Include a screenshot of the schematic and a screenshot of the transient analysis and compare any difference between your pre-lab, simulation, and experimental results in your lab write-up for all five values in each of the three scenarios (including the resistor voltage scenario described below).

Generate a computer simulation modeling the resistor voltage transient of the RLC circuit in Figure 2. Measure the initial resistor voltage, $v_R(0^+)$, the final resistor voltage, $v_R(\infty)$, and the resistor voltage at 0.5 mS, 1 mS and 2 mS. **Record your values in the appropriate sections of Table 2**.

6. Conclusions

This concludes the lab. Make sure to return all components to their appropriate bins. The write-up should include all deliverables and explanations asked for in the procedure as well as the Data Entry and Lab Instructor Signature Page with all recorded numbers, equations, etc.

Data Entry and Lab Instructor Signature Page(s)

Attach this/these page(s) to your write-up.

Table 1: Overdamped RLC Circuit

Prelab Section	Quantity	Calculated Value	Measured, Simulated Value(s)	
A	α		N/A	
A	ω_0		N/A	
A	Type of Damping			
A	$S_{1,2}$		N/A	
A	$v_c(0^+)$			
A	$v_c(\infty)$			
B	$v_c(t) = v_c(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$			
B	$v_c(0^+)$ = $A_1 + A_2$ =			
C	$\dfrac{dv_c(t)}{dt} = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$			
C	$dv_c(0^+)/dt$ = A_1 = A_2 =			
D	$v_c(t)$			
D	$v_c(0.5\ mS)$			
D	$v_c(1\ mS)$			
D	$v_c(2\ mS)$			

Table 2: Underdamped RLC Circuit

<table>
<tr><th>Prelab Section</th><th>Quantity</th><th>Calculated Value</th><th colspan="2">Measured, Simulated Value(s)</th></tr>
<tr><td>E</td><td>α</td><td></td><td colspan="2">N/A</td></tr>
<tr><td>E</td><td>ω_0</td><td></td><td colspan="2">N/A</td></tr>
<tr><td>E</td><td>Type of Damping</td><td></td><td colspan="2"></td></tr>
<tr><td>E</td><td>$S_{1,2}$</td><td></td><td colspan="2">N/A</td></tr>
<tr><td>E</td><td>ω_d</td><td></td><td colspan="2">N/A</td></tr>
<tr><td>E</td><td>$v_c(0^+)$</td><td></td><td></td><td></td></tr>
<tr><td>E</td><td>$v_c(\infty)$</td><td></td><td></td><td></td></tr>
<tr><td>F</td><td colspan="4">$$v_c(t) = v_c(\infty) + (B_1 \cos \omega_d t + B_2 \sin \omega_d t)e^{-\alpha t}$$</td></tr>
<tr><td>F</td><td colspan="4">$v_c(0^+)$ =

B_1 =</td></tr>
<tr><td>G</td><td colspan="4">$$\frac{dv_c(t)}{dt} = (B_1 \cos \omega_d t + B_2 \sin \omega_d t)(-\alpha e^{-\alpha t}) + (\omega_d B_1 \sin \omega_d t + \omega_d B_2 \cos \omega_d t)e^{-\alpha t}$$</td></tr>
<tr><td>G</td><td colspan="4">$dv_c(0^+)/dt$ =

B_1 = B_2 =</td></tr>
<tr><td>H</td><td>$v_c(t)$</td><td colspan="3"></td></tr>
<tr><td>H</td><td>v_c(0.5 mS)</td><td></td><td></td><td></td></tr>
<tr><td>H</td><td>v_c(1 mS)</td><td></td><td></td><td></td></tr>
<tr><td>H</td><td>v_c(2 mS)</td><td></td><td></td><td></td></tr>
</table>

Table 3: Underdamped RLC Circuit, Resistor Voltage

Prelab Section	Quantity	Calculated Value(s)		Measured, Simulated Value(s)	
I	α, ω_0			N/A	
I	Type of Damping				
I	$S_{1,2}$			N/A	
I	ω_d			N/A	
I	$v_R(0^+)$				
I	$v_R(\infty)$				
J	$i_L(t) = i_R(t) = i_L(\infty) + (C_1 \cos \omega_d t + C_2 \sin \omega_d t)e^{-\alpha t}$				
J	$i_L(0)$ = C_1 =				
K	$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = (C_1 \cos 0 + C_2 \sin 0)(-\alpha e^0) + (\omega_d C_1 \sin 0 + \omega_d C_2 \cos 0)e^0$				
K	$\frac{v_L(0^+)}{L} =$ C_1 =		C_2 =		
L	$v_R(t)$				
L	v_R(0.5 mS)				
L	v_R(1 mS)				
L	v_R(2 mS)				

Prelab Signature ____________________ **Experimental Signature** ________________________