Chapter 2

Motion Along a Straight Line

2.1 Displacement, Time, and Average Velocity

1D motion. Very often it is convenient to model an object whose motion you analyze (e.g. car, runner, stone, etc.) as a point particle. Then to describe motion of the object we can use a vector in some coordinate system. If the motion is along a straight line, then it is convenient (to choose x-axis of the coordinate system) to lie along the direction of motion. The origin may be chosen at the location of the object in the initial position, or anywhere along the line of motion.

Position. As time progresses the position vector (or location of the object) will change with time. Thus, position vector is nothing but three functions of time

$$\mathbf{r}(t) = (x(t), y(t), z(t)).$$  \hfill (2.1)

If motion is only along x-axis, then the position is only a single function of time

$$\mathbf{r}(t) = (x(t), 0, 0).$$  \hfill (2.2)

In this chapter we are only interested in motion along a straight line and thus the position will be denoted by $x(t)$. For example,

$$x(t) = 7 \text{ m},$$
$$x(t) = \sin \left( \frac{t}{18} \right) 2 \text{ m} + 5 \text{ m},$$
$$x(t) = t \cdot \left( \frac{5 \text{ m}}{s} \right)$$
$$x(t) = t^2 \cdot \left( \frac{3 \text{ m}}{s^2} \right) + t \cdot \left( \frac{4 \text{ m}}{s} \right) + 5 \text{ m}$$  \hfill (2.3)
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Average velocity. Whatever the function \( x(t) \) is one can always find displacement

\[
\Delta x \equiv x(t_2) - x(t_1)
\]

during time interval

\[
\Delta t \equiv t_2 - t_1
\]

and average velocity as

\[
v_{\text{avg}} \equiv \frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.
\]

So if you know the positions of your object \( x(t_1) \) and \( x(t_2) \) at two different moments of time \( t_1 \) and \( t_2 \), then you can always calculate average velocity. For example, if a runner finished a 10 km race in 1 hour, then his/her average velocity was

\[
v_{\text{avg}} = \frac{x(1h) - x(0h)}{1h - 0h} = \frac{10\text{km}}{1h} = 10\text{km/h}.
\]

Since \( \Delta t \) is always positive, the displacement \( \Delta x \) and thus average velocity can be

\[
v_{\text{avg}} \begin{cases} 
> 0 & \text{motion mostly in the direction of } + \hat{i} \\
< 0 & \text{motion mostly in the direction of } - \hat{i} \\
= 0 & \text{for any round-trip motion.}
\end{cases}
\]

Question: Can traveled distance be larger than displacement? Can it be smaller? Can it be the same?

2.2 Instantaneous Velocity

Instantaneous velocity. Average velocity is telling you how on average the position of your object is changing during time interval \( \Delta t \). Instantaneous velocity is the same thing, but when the time interval is taken to zero, i.e.

\[
v(t) \equiv \lim_{\Delta t \to 0} v_{\text{avg}} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}.
\]
Thus given $x$ as a function of $t$ one can always find instantaneous velocity at every moment of time by simple differentiation. For example,

$$x(t) = 7 \text{ m}, \quad \rightarrow \quad v(t) = \frac{dx}{dt} = 0 \text{ m/s}$$

$$x(t) = \sin \left(\frac{t}{1\text{s}}\right) 2 \text{ m} + 5 \text{ m} \quad \rightarrow \quad v(t) = \frac{dx}{dt} = \cos \left(\frac{t}{1\text{s}}\right) 2 \text{ m/s}$$

$$x(t) = t \cdot \left(\frac{3\text{ m}}{s}\right) \quad \rightarrow \quad v(t) = \frac{dx}{dt} = 5 \text{ m/s}$$

Graphical representation of both types of velocities should be clear: average velocity is a slope of a line joining the coordinates of initial $(x_1, t_1)$ and final $(x_2, t_2)$ points points on $x(t)$ graph; instantaneous velocity if a slope of the tangent line to $x(t)$ at a given time.

### 2.3 Average and Instantaneous acceleration

**Average acceleration.** Given instantaneous velocity as a function of time $v(t)$, one can calculate average acceleration.

$$a_{\text{avg}} \equiv \frac{\Delta v}{\Delta t} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}. \quad (2.10)$$

This is analogous to how the average velocity was defined from (instantaneous) position. For example if a car was going with velocity $v(t_1) = 100 \text{ km/h}$ and then one hour later its velocity is $150 \text{ km/h}$, then average acceleration must have been

$$a_{\text{avg}} = \frac{150 \text{ km/h} - 100 \text{ km/h}}{1 \text{ h}} = 50 \text{ km/h}^2. \quad (2.11)$$

This is completely independent on what the instantaneous velocity was during this hour.

**Instantaneous acceleration.** By drawing an analogy with instantaneous velocity, one might guess that instantaneous acceleration is defined by taking a limit,

$$a(t) \equiv \lim_{\Delta t \to 0} a_{\text{avg}} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}. \quad (2.12)$$

Thus instantaneous acceleration is an acceleration at a given moment of time.
For example,

\[ v(t) = 0 \text{ m/s}, \quad \Rightarrow \quad a(t) = \frac{dv}{dt} = 0 \text{ m/s}^2 \]

\[ v(t) = \cos \left( \frac{t}{1 \text{s}} \right) \cdot 2 \text{ m/s} \quad \Rightarrow \quad a(t) = \frac{dv}{dt} = -\sin \left( \frac{t}{1 \text{s}} \right) \cdot 2 \text{ m/s}^2 \]

\[ v(t) = 5 \text{ m/s} \quad \Rightarrow \quad a(t) = \frac{dv}{dt} = 0 \text{ m/s}^2 \]

\[ v(t) = t \cdot \left( 6 \frac{\text{m}}{\text{s}^2} \right) + 4 \frac{\text{m}}{\text{s}} \quad \Rightarrow \quad a(t) = \frac{dv}{dt} = 6 \frac{\text{m}}{\text{s}^2} \quad (2.13) \]

Graphical representation of both types of accelerations should be clear: average velocity is a slope of a line joining the coordinates of initial \((v(t_1), t_1)\) and final \((v(t_2), t_2)\) points on \(v(t)\) graph; instantaneous velocity if a slope of the tangent line to \(v(t)\) at a given time.

### 2.4 Motion with Constant Acceleration

**Constant acceleration.** Motion with constant acceleration is a motion for which instantaneous acceleration is a constant function

\[ a(t) = a_x, \quad (2.14) \]

where the subscript \(x\) is to remind you that the motion is along \(x\) axis only. Velocity for such motion changes as

\[ v_x(t) = v_{0x} + a_x t \quad (2.15) \]

and position changes as

\[ x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2. \quad (2.16) \]

Note that \(a_x, v_{0x}, x_0\) are some fixed numbers representing acceleration, velocity and position at time \(t = 0\), but \(t\) is a variable which can take any non-negative value. The fact that Eqs. (2.14), (2.15) and (2.16) are self-consistent can be checked by differentiation,

\[ v(t) = \frac{dx(t)}{dt} = v_{0x} + a_x t \]

\[ a(t) = \frac{dv(t)}{dt} = a_x. \quad (2.17) \]
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Useful relations. One can obtain other useful relations from Eqs. (2.15) and (2.16) by expressing \( t \) in one equation and plugging it into another. For example, Eq. (2.15) implies

\[
t = \frac{v_x(t) - v_{0x}}{a_x}
\]

and by plugging it in Eq. (2.16) we get

\[
x(t) = x_0 + v_{0x} \left( \frac{v_x(t) - v_{0x}}{a_x} \right) + \frac{1}{2} a_x \left( \frac{v_x(t) - v_{0x}}{a_x} \right)^2
\]

\[
x(t) - x_0 = \frac{v_x(t) \cdot v_{0x} - v_{0x}^2}{a_x} + \frac{v_x(t)^2 - 2v_x(t)v_{0x} + v_{0x}^2}{2a_x}
\]

\[
2a_x (x(t) - x_0) = 2v_x(t) \cdot v_{0x} - 2v_{0x}^2 + v_x(t)^2 - 2v_x(t)v_{0x} + v_{0x}^2
\]

\[
2a_x (x(t) - x_0) = v_x(t)^2 - v_{0x}^2
\]

or

\[
v_x(t)^2 = v_{0x}^2 + 2a_x (x(t) - x_0).
\]

Another useful relation is obtained from Eqs. (2.15) and (2.16) by expressing \( a_x \) in one equation and plugging it into another. For example, Eq. (2.15) implies

\[
a_x = \frac{v_x(t) - v_{0x}}{t}
\]

and by plugging it in Eq. (2.16) we get

\[
x(t) = x_0 + v_{0x} t + \frac{1}{2} \left( \frac{v_x(t) - v_{0x}}{t} \right) t^2
\]

\[
x(t) = x_0 + v_{0x} t + \frac{1}{2} v_x(t) t - \frac{1}{2} v_{0x} t
\]

\[
x(t) - x_0 = \frac{1}{2} v_x(t) t + \frac{1}{2} v_{0x} t
\]

or

\[
x(t) - x_0 = \frac{1}{2} (v_x(t) + v_{0x}) t.
\]

Question: What would change in equations (2.14), (2.15), (2.16), (2.20) and (2.23) if the initial time is at \( t_0 \neq 0 \)?

Example 2.4. A motorcyclist heading east through a small town accelerates at a constant acceleration 4.0 m/s\(^2\) after he leaves city limits. At time \( t = 0 \) he is 5.0 m east of the city-limits while he moves east at 15 m/s. \( a \)

Find position and velocity at \( t = 2.0 \) s. \( b \) Where is he when his speed is 25 m/s?
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Step 1: Coordinate system: Where should the x-axis point? Pointing east with origin at the city-limit.

Step 2: Initial conditions: What is known about initial conditions?

\[
t_0 = 0 \text{ s} \\
x_0 = 5.0 \text{ m} \\
v_{0x} = 15 \text{ m/s} \\
a_x = 4.0 \text{ m/s}^2.
\]  
(2.24)

Step 3: Final conditions: (a) What is position and velocity at \( t = 2.0 \text{ s} \)?

Using Eq. (2.16) we get

\[
x(2.0 \text{ s}) = (5.0 \text{ m}) + (15 \text{ m/s})(2.0 \text{ s}) + \frac{1}{2} \cdot 4.0 \text{ m/s}^2 (2.0 \text{ s})^2 = 43 \text{ m}
\]  
(2.25)

and using Eq. (2.15)

\[
v_x(2.0 \text{ s}) = 15 \text{ m/s} + (4.0 \text{ m/s}^2)(2.0 \text{ s}) = 23 \text{ m/s}.
\]  
(2.26)

(b) Where is he when his speed is 25 m/s? Using Eq. (2.20) we get

\[
x(t) = x_0 + \frac{v_x(t)^2 - v_{0x}^2}{2a_x}
\]  
(2.27)

or by substituting the known quantities

\[
x(t) = (5.0 \text{ m}) + \frac{(25 \text{ m/s})^2 - (15 \text{ m/s})^2}{2 \cdot (4.0 \text{ m/s}^2)} = 55 \text{ m}.
\]  
(2.28)

2.5 Free Falling Bodies

Free falling. Free fall is a very deep concept in physics. All it means that in 4-dimensional space-time free falling object move along straight lines (geodesics). For now we are only interested in motion very close to surface of
the Earth and such motions can be approximated as motions with constant acceleration

\[ a_x = g = 9.80 \text{ m/s}^2. \]  \hspace{1cm} (2.29)

Thus all of the concepts and equations considered in the previous section apply. Note that \( g \) is taken to be positive and thus it makes sense to choose \( x \)-axis to point vertically and downward. If you choose the \( x \)-axis to point upward then \( a_x = -9.80 \text{ m/s}^2 \).

**Example 2.6.** One-euro coin is dropped from the Leaning Tower of Pisa and falls free from rest. What are its position and velocity after 1.0 s?

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Step 1. Coordinate system: Where should the \( x \)-axis point? Let’s assume that \( x \)-axis points vertically and downwards (this is opposite to what the book assumes) with origin at the top of Pisa tower.
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Step 2. Initial condition: What is known about initial conditions?

\[
\begin{align*}
t_0 & = 0 \text{ s} \\
x_0 & = 0 \text{ m} \\
v_{0x} & = 0 \text{ m/s} \\
a_x & = +9.80 \text{ m/s}^2.
\end{align*}
\] (2.30)

Step 3. Final conditions: What are its position and velocity after 1.0 s?

Using Eq. (2.16) we get

\[
x(1.0 \text{ s}) = (0 \text{ m}) + (0 \text{ m/s})(1.0 \text{ s}) + \frac{1}{2}9.80 \text{ m/s}^2 (1.0 \text{ s})^2 = 4.9 \text{ m}. \] (2.31)

and using Eq. (2.15)

\[
v_x(1.0 \text{ s}) = 0 \text{ m/s} + (+9.80 \text{ m/s}^2)(1.0 \text{ s}) = 9.80 \text{ m/s}. \] (2.32)

Example 2.7. You throw a ball vertically upward from the roof of a tall building. The ball leaves your hand at a point even with the roof railing with an upward speed of 15.0 m/s; the ball is then in free fall. On its way back, it just misses the railing. Find (a) the ball’s position and velocity 1.00 s and 4.00 s after leaving your hand; (b) the ball’s velocity when it is 5.00 m above the railing; (c) the maximum height reached; (d) the ball’s acceleration when it is at its maximum height.
Step 1: Coordinate system: Choose x-axis to point vertically and upward with origin at the roof railing.

Step 2: Initial conditions: What is known about initial conditions?

\[
\begin{align*}
t_0 &= 0 \text{ s} \\
x_0 &= 0 \text{ m} \\
v_{0x} &= 15.0 \text{ m/s} \\
a_x &= -9.80 \text{ m/s}^2.
\end{align*}
\]

(2.33)

Step 3: Final conditions: (a) What is ball’s position and velocity \( t = 1.00 \text{ s} \) and \( t = 4.00 \text{ s} \) after leaving your hand? Using Eq. (2.16) we get

\[
\begin{align*}
x(1.00 \text{ s}) &= (0 \text{ m}) + (15.0 \text{ m/s})(1.0 \text{ s}) + \frac{1}{2} (-9.80 \text{ m/s}^2)(1.0 \text{ s})^2 = 10.1 \text{ m} \\
x(4.00 \text{ s}) &= (0 \text{ m}) + (15.0 \text{ m/s})(4.0 \text{ s}) + \frac{1}{2} (-9.80 \text{ m/s}^2)(4.0 \text{ s})^2 = -18.44 \text{ m}
\end{align*}
\]
and using Eq. (2.15)

\[ v_x(1.0 \text{ s}) = 15.0 \text{ m/s} + (-9.80 \text{ m/s}^2)(1.0 \text{ s}) = 5.2 \text{ m/s} \]
\[ v_x(4.0 \text{ s}) = 15.0 \text{ m/s} + (-9.80 \text{ m/s}^2)(4.0 \text{ s}) = -24.2 \text{ m/s}. \] (2.35)

(b) What is the ball’s velocity when it is 5.00 m above the railing? Using Eq. (2.20)

\[ v_x(t)^2 = v_{0x}^2 + 2a_x(x(t) - x_0). \] (2.36)

we get

\[ v_x = \sqrt{(15.0 \text{ m/s})^2 + 2(-9.80 \text{ m/s}^2)(5.00 \text{ m} - 0 \text{ m})} \] (2.37)

with two solutions

\[ v_x = +11.3 \text{ m/s} \] (2.38)
or

\[ v_x = -11.3 \text{ m/s}. \] (2.39)

(c) What is the maximum height reached? Using Eq. (2.27)

\[ x = (0 \text{ m}) + \left( \frac{(0 \text{ m/s})^2 - (15.0 \text{ m/s})^2}{2(-9.80 \text{ m/s}^2)} \right) = 11.5 \text{ m}. \] (2.40)

(d) What is the ball’s acceleration when it is at its maximum height? We are considering motion with constant acceleration and thus the acceleration is always

\[ a_x = -9.80 \text{ m/s}^2 \] (2.41)

where the minus sign (for our choice of coordinate system, see Step 1) indicates that the acceleration points downwards.

### 2.6 Velocity and Position by Integration.

**Velocity by integration.** An expression for velocity can always be obtain given acceleration as a function of time. From definition of acceleration

\[ a_x(t) = \frac{dv_x(t)}{dt} \] (2.42)

and thus in differential from

\[ dv_x = a_x(t)dt \] (2.43)

or in integral form

\[ \int_{v_x(t_1)}^{v_x(t_2)} dv_x = \int_{t_1}^{t_2} a_x(t)dt. \] (2.44)
Therefore for \( t_1 = 0 \) and \( t_2 = T \) we have

\[
v_x(T) = v_x(0) + \int_0^T a_x(t)dt. \tag{2.45}
\]

If we replace

- \( T \rightarrow t \)
- \( t \rightarrow \tau \)
- \( v_x(0) \rightarrow v_{0x} \) \tag{2.46}

then we get

\[
v_x(t) = v_{0x} + \int_0^t a_x(\tau)d\tau \tag{2.47}
\]

and in the case of constant acceleration

\[
v_x(t) = v_{0x} + a_x t. \tag{2.48}
\]

**Position by integration.** Similarly, an expression for position can be obtain given velocity as a function of time. From definition of velocity

\[
v_x(t) = \frac{dx(t)}{dt} \tag{2.49}
\]

and thus in differential from

\[
dx = v_x(t)dt \tag{2.50}
\]

or in integral form

\[
\int_{x(t_1)}^{x(t_2)} dx = \int_{t_1}^{t_2} v_x(t)dt. \tag{2.51}
\]

Therefore for \( t_1 = 0 \) and \( t_2 = T \) we have

\[
x(T) = x(0) + \int_0^T v_x(t)dt. \tag{2.52}
\]

If we replace

- \( T \rightarrow t \)
- \( t \rightarrow \tau \)
- \( x(0) \rightarrow x_0 \) \tag{2.53}
then we get
\[ x(t) = x_0 + \int_0^t v_x(\tau) d\tau \] (2.54)
and in the case of constant acceleration
\[
\begin{align*}
x(t) &= x_0 + \int_0^t (v_{0x} + a_x \tau) d\tau \\
x(t) &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2.
\end{align*}
\] (2.55)

**Example 2.9.** Sally is driving along a straight highway in her 1965 Mustang. At \( t = 0 \), when she is moving at 10 m/s in the positive \( x \)-direction, she passes a signpost at \( x = 50 \) m. Her \( x \)-acceleration as a function of time is
\[ a_x(t) = 2.0 \text{ m/s}^2 - (0.10 \text{ m/s}^3) t. \] (2.56)

(a) Find her \( x \)-velocity \( v_x(t) \) and \( x(t) \) as functions of time. (b) When is her \( x \)-velocity greatest? (c) What is that maximum \( x \)-velocity? (d) Where is the car when it reaches that maximum \( x \)-velocity?

Step 1: Coordinate system: The coordinate system was already chosen for us with \( x \)-axis pointing along the direction of motion and origin at \( x = 0 \) m signpost.

Step 2: Initial conditions: What is known about initial conditions?
\[
\begin{align*}
t_0 &= 0 \\
x_0 &= 50 \text{ m} \\
v_{0x} &= 10 \text{ m/s} \\
a_{0x} &= 2.0 \text{ m/s}^2.
\end{align*}
\] (2.57)

Step 3: Final conditions: (a) What is her \( x \)-velocity and position as a
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function of time? By substituting Eq. (2.56) into Eq. (2.44) we get

\[ v_x(t) = (10 \text{ m/s}) + \int_{0}^{t} (2.0 \text{ m/s}^2 - (0.10 \text{ m/s}^3) \tau) d\tau \]

\[ v_x(t) = 10 \text{ m/s} + t \cdot 2.0 \text{ m/s}^2 - \frac{1}{2} t^2 \cdot 0.10 \text{ m/s}^3 \]

\[ v_x(t) = 10 \text{ m/s} + t \cdot 2.0 \text{ m/s}^2 - t^2 \cdot 0.05 \text{ m/s}^3 \] (2.58)

and from Eq. (2.54) we get

\[ x(t) = (50 \text{ m}) + \int_{0}^{t} (10 \text{ m/s} + t \cdot 2.0 \text{ m/s}^2 - t^2 \cdot 0.05 \text{ m/s}^3) d\tau \]

\[ x(t) = 50 \text{ m} + t \cdot 10 \text{ m/s} + \frac{1}{2} t^2 \cdot 2.0 \text{ m/s}^2 - \frac{1}{3} t^3 \cdot 0.05 \text{ m/s}^3 \]

\[ x(t) \approx 50 \text{ m} + t \cdot 10 \text{ m/s} + t^2 \cdot 1.0 \text{ m/s}^2 - t^3 \cdot 0.017 \text{ m/s}^3 \] (2.59)

(b) When is her \( x \)-velocity greatest? The maximum of a function occurs if the slope changes from positive to negative. This happens in an instance when the slope is exactly zero, i.e.

\[ \frac{dv_x(t_{\text{max}})}{dt} = a_x(t_{\text{max}}) = 2.0 \text{ m/s}^2 - t_{\text{max}} \cdot 0.10 \text{ m/s}^3 = 0 \] (2.60)

or when

\[ t_{\text{max}} = \frac{2.0 \text{ m/s}^2}{0.10 \text{ m/s}^3} = 20 \text{ s.} \] (2.61)

(c) What is that maximum \( x \)-velocity? Using Eq. (2.58) we get

\[ v_x(20 \text{ s}) = 10 \text{ m/s} + 20 \text{ s} \cdot 2.0 \text{ m/s}^2 - (20 \text{ s})^2 \cdot 0.05 \text{ m/s}^3 = 30 \text{ m/s.} \] (2.62)

(d) Where is the car when it reaches that maximum \( x \)-velocity? Using Eq. (2.59) we get

\[ x(20 \text{ s}) = 50 \text{ m} + (20 \text{ s}) \cdot 10 \text{ m/s} + (20 \text{ s})^2 \cdot 1.0 \text{ m/s}^2 - (20 \text{ s})^3 \cdot 0.017 \text{ m/s}^3 = 517 \text{ m.} \] (2.63)