Chapter 3

Motion in Two or Three dimensions

3.1 Position and Velocity Vectors

Extra dimensions. We now generalize the results of previous section to motion in more than one (spacial) dimension. In this chapter we will only concentrate on motions in two and three dimensions (often abbreviated as 2D and 3D) which is what we typically observe by naked eye. However, this does not mean that at very large (cosmological) or at very small (high energy) scales the effect of other (usually called extra) dimensions are not present. In fact there are very strong arguments supporting the ideas of extra dimensions base on string theory. The reason the extra dimensions (usually six and compact) had to be introduced in such theories is in an attempt to make the physical theories well defined mathematically, but we are still far from accomplishing this task. However, even if there are extra dimensions, but the motion along those extra dimensions is small, then we need not worry about them. This is similar how we did not have to worry about all three dimensions in the previous chapter by considering motion along only a single axis.

Position. Position vector $\vec{r}$ specifies position of an object in three dimensions and in a given coordinate systems it it is described by three numbers,

$$\vec{r} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$$  \hspace{1cm} (3.1)

One can imagine an arrow connecting the origin with position of object to describe $\vec{r}$. If the object is in motion, then the position changes with time the position vector becomes a function (actually three functions) of time

$$\vec{r}(t) = (x(t), y(t), z(t)) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$  \hspace{1cm} (3.2)
These three functions do depend on the coordinate system, but as was already noted the physically measurable quantities should not depend on this choice.

**Velocity.** Similarly to 1D case (see Eqs. (2.5) and (2.8)) in 3D we can define average velocity vector as

\[
\mathbf{v}_{\text{avg}} \equiv \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t_2) - \mathbf{r}(t_1)}{t_2 - t_1},
\]  

(3.3)

(where \(\Delta \mathbf{r}\) is now the displacement vector) and instantaneous velocity vector

\[
\mathbf{v}(t) \equiv \lim_{\Delta t \to 0} \mathbf{v}_{\text{avg}} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}(t)}{dt}.
\]  

(3.4)

Then we can use the components representation of position vector Eq. (3.2) to obtain

\[
\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \left( \frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt} \right) = \frac{dx(t)}{dt} \mathbf{i} + \frac{dy(t)}{dt} \mathbf{j} + \frac{dz(t)}{dt} \mathbf{k}
\]  

(3.5)

and if we denote

\[
\mathbf{v} \equiv (v_x, v_y, v_z)
\]  

(3.6)

then

\[
\begin{align*}
v_x &= \frac{dx(t)}{dt} \\
v_y &= \frac{dy(t)}{dt} \\
v_z &= \frac{dz(t)}{dt}.
\end{align*}
\]  

(3.7)

**Example 3.1.** A robotic vehicle, or rover, is exploring the surface of Mars. The stationary Mars lander is the origin of coordinates, and the surrounding Martian surface lies in the xy-plane. The rover, which we represent as a point, has x- and y-coordinates that vary with time:

\[
\begin{align*}
x(t) &= 2.0 \text{ m} - (0.25 \text{ m/s}^2) t^2 \\
y(t) &= (1.0 \text{ m/s}) t + (0.025 \text{ m/s}^3) t^3 \\
z(t) &= 0 \text{ m}
\end{align*}
\]  

(3.8)

(a) Find the rover’s coordinates and distance from the lander at \(t = 2.0\) s. (b) Find the rover’s displacement and average velocity vectors for the interval \(t = 0.0\) s to \(t = 2.0\) s. (c) Find a general expression for the rover’s instantaneous velocity vector \(\mathbf{v}\). Express \(\mathbf{v}\) at \(t = 2.0\) s in components from and in
terms of magnitude and direction.

Step 1: Coordinate system: Was already chosen for you with origin at Mars lander and Martian surface in the \( xy \)-plane.

Step 2: Initial conditions: Easy to calculate

\[
\vec{r}(0) = (x(0), y(0), z(0)) = (2.0 \text{ m, } 0 \text{ m, } 0 \text{ m})
\]

\[
\vec{v}(0) = \left( \frac{dx(0)}{dt}, \frac{dy(0)}{dt}, \frac{dz(0)}{dt} \right) = (0 \text{ m/s, } 1.0 \text{ m/s, } 0 \text{ m/s}) \quad (3.9)
\]

Step 3: Final conditions: (a) Find the rover’s coordinates and distance from the lander at \( t = 2.0 \text{ s} \). Using Eq. (3.8) we get coordinates

\[
x(2.0 \text{ s}) = 2.0 \text{ m} - (0.25 \text{ m/s}^2) (2.0 \text{ s})^2 = 1.0 \text{ m}
\]

\[
y(2.0 \text{ s}) = (1.0 \text{ m/s}) (2.0 \text{ s}) + (0.025 \text{ m/s}^3) (2.0 \text{ s})^3 = 2.2 \text{ m}
\]

\[
z(t) = 0 \text{ m} \quad (3.10)
\]

and distance from origin

\[
r(2.0 \text{ s}) = |\vec{r}(2.0 \text{ s})| = \sqrt{x(2.0 \text{ s})^2 + y(2.0 \text{ s})^2 + z(2.0 \text{ s})^2} = \sqrt{1^2 + 2.2^2 + 0^2} \approx 2.4 \text{ m}.
\]

(b) Find the rover’s displacement and average velocity vectors for the interval \( t = 0.0 \text{ s} \) to \( t = 2.0 \text{ s} \). Displacement is

\[
\Delta \vec{r} = \vec{r}(2.0 \text{ s}) - \vec{r}(0.0 \text{ s})
\]

\[
= (1.0 \text{ m, } 2.2 \text{ m, } 0 \text{ m}) - (2.0 \text{ m, } 0 \text{ m, } 0 \text{ m})
\]

\[
= (-1.0 \text{ m, } 2.2 \text{ m, } 0 \text{ m}) \quad (3.12)
\]
or
\[ \Delta \mathbf{r} = (-1.0 \text{ m}) \mathbf{i} + (2.2 \text{ m}) \mathbf{j} \]  
(3.13)
and thus, average velocity is
\[ \bar{\mathbf{v}}_{\text{avg}} = \frac{(-1.0 \text{ m}) \mathbf{i} + (2.2 \text{ m}) \mathbf{j}}{2.0 \text{ s}} = (-0.5 \text{ m/s}) \mathbf{i} + (1.1 \text{ m/s}) \mathbf{j}. \]  
(3.14)

(c) Find a general expression for the rover’s instantaneous velocity vector \( \mathbf{v} \). Express \( \mathbf{v} \) at \( t = 2.0 \text{ s} \) in components from and in terms of magnitude and direction. By plugging Eq. (3.8) into Eq. (3.5) we get
\[ v_x(t) = \frac{dx(t)}{dt} = -0.5 \text{ m/s}^2 t \]
\[ v_y(t) = \frac{dy(t)}{dt} = 1.0 \text{ m/s} + (0.075 \text{ m/s}^3) t^2 \]
\[ v_z(t) = \frac{dz(t)}{dt} = 0 \text{ m/s} \]  
(3.15)
and at \( t = 2.0 \text{ s} \)
\[ v_x(2.0 \text{ s}) = \frac{dx(2.0 \text{ s})}{dt} = -0.5 \text{ m/s}^2 (2.0 \text{ s}) = -1.0 \text{ m/s} \]
\[ v_y(2.0 \text{ s}) = \frac{dy(2.0 \text{ s})}{dt} = 1.0 \text{ m/s} + (0.075 \text{ m/s}^3) (2.0 \text{ s})^2 = 1.3 \text{ m/s} \]
\[ v_z(2.0 \text{ s}) = \frac{dz(2.0 \text{ s})}{dt} = 0 \text{ m/s}. \]  
(3.16)
or
\[ \mathbf{v}(2.0 \text{ s}) = (-1.0 \text{ m/s}) \mathbf{i} + (1.3 \text{ m/s}) \mathbf{j}. \]  
(3.17)
Then the magnitude of velocity (or speed) is
\[ v = |\mathbf{v}| = \sqrt{(-1.0 \text{ m/s})^2 + (1.3 \text{ m/s})^2} = 1.6 \text{ m/s} \]  
(3.18)
and direction is
\[ \arctan \frac{v_y}{v_x} = \arctan \frac{1.3 \text{ m/s}}{-1.0 \text{ m/s}} = -52^\circ \]  
(3.19)
which is off by 180° because of the way arctan function is defined (range is from \(-90^\circ\) to \(+90^\circ\)). Thus the direction (measure from x-axis in counterclockwise direction) is
\[ \alpha = 180^\circ - 52^\circ = 128^\circ. \]  
(3.20)
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3.2 The Acceleration Vector

**Acceleration.** Similarly to 1D case (see Eqs. (2.10) and (2.12)) in 3D we can define average acceleration vector as

$\mathbf{a}_{\text{avg}} \equiv \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}(t_2) - \mathbf{v}(t_1)}{t_2 - t_1},$ \hspace{1cm} (3.21)

and instantaneous velocity vector

$\mathbf{a}(t) \equiv \lim_{\Delta t \to 0} \mathbf{a}_{\text{avg}} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}(t)}{dt}.$ \hspace{1cm} (3.22)

Then we can use the components representation of velocity vector Eq. (3.5) to obtain

$\mathbf{a} = \frac{d\mathbf{v}(t)}{dt} = \left( \frac{dv_x(t)}{dt}, \frac{dv_y(t)}{dt}, \frac{dv_z(t)}{dt} \right) = \frac{dv_x(t)}{dt} \mathbf{i} + \frac{dv_y(t)}{dt} \mathbf{j} + \frac{dv_z(t)}{dt} \mathbf{k}$ \hspace{1cm} (3.23)

and if we denote

$\mathbf{\ddot{a}} \equiv (a_x, a_y, a_z)$ \hspace{1cm} (3.24)

then using Eq. (3.7)

$a_x = \frac{dv_x(t)}{dt} = \frac{d^2x(t)}{dt^2}$

$a_y = \frac{dv_y(t)}{dt} = \frac{d^2y(t)}{dt^2}$

$a_z = \frac{dv_z(t)}{dt} = \frac{d^2z(t)}{dt^2}. \hspace{1cm} (3.25)$

**Example 3.2.** Let’s return to motion of the Mars from the previous section.

(a) Find the components of the average acceleration for the interval $t = 0.0$ s to $t = 2.0$ s. (b) Find the instantaneous acceleration at $t = 2.0$ s.
Step 1: Coordinate system: As before the coordinate system was chosen for us.

Step 2: Initial conditions: In addition to initial conditions for position and velocity (see Eq. (3.9)), we can now calculate the initial conditions for acceleration

\[
\vec{a}(0) = \left( \frac{d^2x(0)}{dt^2}, \frac{d^2y(0)}{dt^2}, \frac{d^2z(0)}{dt^2} \right) = (-0.5 \text{ m/s}^2, 0 \text{ m/s}^2, 0 \text{ m/s}) \quad (3.26)
\]

Step 3: Final conditions: (a) Find the components of the average acceleration for the interval \( t = 0.0 \text{ s} \) to \( t = 2.0 \text{ s} \). By substituting initial and final velocities from Eq. (3.9) and (3.16) into Eq. (3.21) we get

\[
\vec{a}_{\text{avg}} = \frac{\vec{v}(2.0 \text{ s}) - \vec{v}(0.0 \text{ s})}{2.0 \text{ s} - 0.0 \text{ s}} = \frac{(-1.0 \text{ m/s}, 1.3 \text{ m/s}, 0 \text{ m/s}) - (0 \text{ m/s}, 1.0 \text{ m/s}, 0 \text{ m/s})}{2.0 \text{ s}} = (-0.5 \text{ m/s}^2, 0.15 \text{ m/s}^2, 0 \text{ m/s}^2) \quad (3.27)
\]

or

\[
\vec{a}_{\text{avg}} = (-0.5 \text{ m/s}^2) \hat{i} + (0.15 \text{ m/s}^2) \hat{j}. \quad (3.28)
\]

(b) Find the instantaneous acceleration at \( t = 2.0 \text{ s} \). Using the definition of Eq. (3.23) and instantaneous velocity that was already calculated above (see
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Eq. (3.15) we get a general expression

\[
\vec{a}(t) = \frac{dv_x(t)}{dt}\hat{i} + \frac{dv_y(t)}{dt}\hat{j} + \frac{dv_z(t)}{dt}\hat{k},
\]

\[
= (-0.5 \text{ m/s}^2)\hat{i} + (0.15 \text{ m/s}^2) t\hat{k}
\]

(3.29)

and at a specific time \( t = 2.0 \text{ s} \)

\[
\vec{a}(2.0 \text{ s}) = (-0.5 \text{ m/s}^2)\hat{i} + (0.15 \text{ m/s}^3) (2.0)\hat{k}
\]

\[
= (-0.5 \text{ m/s}^2)\hat{i} + (0.3 \text{ m/s}^2)\hat{k}.
\]

(3.30)

**Parallel and Perpendicular Components.** It is sometimes useful to think of components of a vector not in a fixed coordinate system but with respect to changing coordinate system. A particularly useful example if for a moving coordinate system in 2D when one of the axis point in the direction of the velocity vector \( \vec{v} \). Then we can decompose the acceleration vector into component parallel \( \vec{a}_|| \) and perpendicular \( \vec{a}_\perp \) to the direction of motion

\[
\vec{a} = \vec{a}_|| + \vec{a}_\perp.
\]

(3.31)

After a little bit of calculus one can show that the magnitude of velocity vector changes as

\[
\frac{dv(t)}{dt} = \frac{d}{dt} \sqrt{\vec{v}(t) \cdot \vec{v}(t)} = \frac{\vec{v}(t)}{v(t)} \cdot \frac{d\vec{v}(t)}{dt} = \vec{v}(t) \cdot \left( \vec{a}_|| (t) + \vec{a}_\perp (t) \right) = \vec{v}(t) \cdot \vec{a}_|| (t)
\]

(3.32)

and the direction of the velocity vector changes as

\[
\frac{d\vec{v}(t)}{dt} = \frac{d}{dt} \left( \frac{\vec{v}(t)}{v(t)} \right) = \frac{\frac{d\vec{v}(t)}{dt} v(t) - \frac{dv(t)}{dt} \vec{v}(t)}{v(t)^2} = \frac{(\vec{a}_|| (t) + \vec{a}_\perp (t)) v(t) - (\vec{v}(t) \cdot \vec{a}_|| (t)) \vec{v}(t)}{v(t)^2} = \vec{a}_\perp (t) \frac{\vec{v}(t)}{v(t)}.
\]

(3.33)

Evidently, the parallel component is responsible for changes in magnitude of the velocity vector (i.e. speed), but not the direction

\[
\vec{a}_|| = \begin{cases} \frac{a||}{v} \vec{v} & \text{speed is increasing} \\ 0 & \text{speed is not changing} \\ -\frac{a||}{v} \vec{v} & \text{speed is decreasing}. \end{cases}
\]

(3.34)

and the perpendicular component is responsible for changes in direction of velocity, but not in magnitude. If there is only a perpendicular component then the object is in circular motion.
3.3 Projectile Motion

**Projectile motion**: is a motion which is completely determined by gravitationally acceleration and air resistance starting from some initial condition determined by position and velocity. (The fact that you have to always specify an even number of initial data is a consequence of the fact that equations of motion are (almost always) of the second order in time.) Such motion is always confined to a 2D plane determined by two vectors: gravitational acceleration vector and initial velocity vector. It is convenient to choose a coordinate system so that one of the axis (usually \( y\)-axis pointing upward) is vertical, the other one (usually \( x\)-axis) is horizontal and since there is no motion in \( z\)-direction it does not really matter. Then,

\[
\mathbf{a} = (0, -g) = -g \mathbf{j} \tag{3.35}
\]

Then from the equations of motion we get

\[
\mathbf{v}(t) = v_{0x} \mathbf{i} + (v_{0y} - gt) \mathbf{j}
\]

\[
\mathbf{r}(t) = (x_0 + v_{0x} t) \mathbf{i} + (y_0 + v_{0y} t - \frac{1}{2}gt^2) \mathbf{j} \tag{3.36}
\]

**Example 3.6.** A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal with magnitude 9.0 m/s. Find the motorcycle’s position, distance from the edge of the cliff, and velocity 0.50 s after it leaves the edge of the cliff.

Step 1: Coordinate System: Let's choose \( y\)-axis to point vertically and upward, and \( x\)-axis to point horizontally in the direction of the motorcycle’s motion before it rides off the cliff. The origin can be at the edge of the cliff.

Step 2: Initial Conditions: The origin position and velocity is given by

\[
\mathbf{r}(0) = (x_0, y_0) = (0, 0)
\]

\[
\mathbf{v}(0) = (v_{0x}, v_{0y}) = (9.0 \text{ m/s}, 0 \text{ m/s}). \tag{3.37}
\]
Step 3: Final Conditions: Find the motorcycle’s position, distance from the edge of the cliff, and velocity 0.50 s after it leaves the edge of the cliff. Using Eq. (3.36) we get velocity and position
\[
\vec{v}(0.50 \text{ s}) = (9.0 \text{ m/s}) \hat{i} + (-9.8 \text{ m/s}^2 \cdot 0.50 \text{ s}) \hat{j} = (9.0 \text{ m/s}) \hat{i} - (4.9 \text{ m/s}) \hat{j}
\]
\[
\vec{r}(0.50 \text{ s}) = (9.0 \text{ m/s} \cdot 0.50 \text{ s}) \hat{i} + \left(-\frac{1}{2}9.8 \text{ m/s}^2 (0.50 \text{ s})^2\right) \hat{j} = (4.5 \text{ m}) \hat{i} + (-12.25 \text{ m}) \hat{j}
\]
and thus the distance from the edge is
\[
-r(0.50 \text{ s}) = \sqrt{(4.5 \text{ m})^2 + (-1.2 \text{ m})^2} = 4.7 \text{ m}. \quad (3.39)
\]

### 3.4 Motion in a Circle

**Uniform motion** - is a motion when the direction of velocity vector changes, but magnitude (or speed) does not change, i.e.
\[
\frac{dv}{dt} = 0 \\
\frac{d\vec{v}}{dt} \neq 0, \quad (3.40)
\]
This already implies that
\[
\vec{v} \cdot \frac{d\vec{v}}{dt} = 0 \quad (3.41)
\]
but if the magnitude of acceleration also does not change,
\[
a(t) = a_\perp(t) = \text{const.} \quad (3.42)
\]
then
\[
0 = \frac{d}{dt} a^2 = \frac{d}{dt} (\vec{a} \cdot \vec{a}) = \frac{d}{dt} \left( \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt} \right) = 2 \frac{d^2\vec{v}}{dt^2} \cdot \frac{d\vec{v}}{dt} \quad (3.43)
\]
By combining Eqs. (3.41) and (3.43) in 2D we must conclude that
\[
\vec{v} \propto \frac{d^2\vec{v}}{dt^2}, \quad (3.44)
\]
whose solutions are sines and cosines (you will see this differential equation over and over in physics courses).

**Radial (or centripetal) acceleration.** With little more algebra (that we shall skip) one can show the such motion gives rise to circular motion described by equation
\[
R = \sqrt{(x(t) - X)^2 + (y(t) - Y)^2} \quad (3.45)
\]
where \((X, Y)\) is at the center of the circle and \(R\) is its radius. By choosing the origin of coordinates at the center of circle, i.e. making

\[(X, Y) = (0, 0)\] (3.46)

we can simplify Eq. (3.45) to a simple form

\[r = \sqrt{x(t)^2 + y(t)^2} = R\] (3.47)

By differentiating it with respect to time once we get

\[0 = \frac{dx(t)}{dt} x(t) + \frac{dy(t)}{dt} y(t)\] (3.48)

or

\[\vec{r} \cdot \vec{v} = 0\] (3.49)

and by differentiating it twice we get

\[0 = \left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{d^2x(t)}{dt^2}\right) x(t) + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{d^2y(t)}{dt^2}\right) y(t)\] (3.50)

or

\[v^2 = -\vec{a} \cdot \vec{r} = -\vec{a}_\perp \cdot \vec{r}\.] (3.51)

But since the two vectors \(\vec{a}_\perp\) and \(\vec{r}\) always point in opposite directions we have

\[v^2 = a_\perp R\] (3.52)

or

\[a_\perp = \frac{v^2}{R}\.] (3.53)

**Periodic motion.** Since the motion is circular the object must come to were it started at some finite time. This time is called period, \(T\), and the motion is called also periodic. To understand what the period is it is useful to write an exact solution for periodic motion around origin

\[\vec{r}(t) = \left(\begin{array}{c} R \sin \left(2\pi \frac{t}{T} + \phi\right) \\ R \cos \left(2\pi \frac{t}{T} + \phi\right) \end{array}\right)\] (3.54)

where \(R, T\) and \(\phi\) are some constants. (Check for yourself that this solution indeed satisfies Eq. (3.47).) We already discussed \(R\) as it describes the radius of motion, but the other two constants have special meanings as well. Note that as time goes from \(t\) to \(t + T\) the arguments of the sin and cos
functions change by $2\pi$ and thus the values of these functions as well as the position vector does not change

$$\mathbf{r}(t) = \mathbf{r}(t + T).$$  \hspace{1cm} (3.55)

Do not worry about constant $\phi$ for now (it is known as phase). It is now straightforward to figure out the dependance of velocity on the period, which we could have done even without writing the explicit solution. Evidently, an object in circular motion is traveling a distance $2\pi R$ with a constant speed $v$ and thus the period must be

$$T = \frac{2\pi R}{v}$$  \hspace{1cm} (3.56)

or

$$v = \frac{2\pi R}{T}.$$  \hspace{1cm} (3.57)

By combining Eqs. (3.53) and (3.55) we get

$$a_\perp = \frac{4\pi^2 R}{T^2}.$$  \hspace{1cm} (3.58)

**Example 3.12.** Passengers on a carnival ride move at a constant speed in a horizontal circle of radius 5.0 m, making a complete circle in 4.0 s. What is their acceleration?

Step 1: What is the coordinate system? Let choose a (moving) coordinate system with $||$ direction in the direction of motion and $\perp$ direction in the radial direction.

Step 2: What is given? Period $T = 4.0$ s and radius $R = 5.0$ m.

Step 3: What do we have to find? Then according to (3.58) we have

$$a_\perp = \frac{4\pi^2 (5.0 \text{ m})}{(4.0 \text{ s})^2} = 12 \text{ m/s}^2 = 1.3 g$$

and since the motion is uniform

$$a_{||} = 0.$$
Non-uniform motion. If an object is in circular motion, but the magnitude of its velocity (or speed) changes, then such motion is called a nonuniform circular motion. Then in addition to perpendicular (or centripetal) acceleration there is a parallel (or tangential) acceleration.

\[
a_{\perp}(t) = \frac{v(t)^2}{R}
\]

\[
a_{||}(t) = \frac{dv(t)}{dt}.
\]\[3.59\]

3.5 Relative velocity

Reference frame. We have already mentioned how the choice of coordinate system is important for calculations, but should not matter for physically observable quantities. We have also discussed (in context of 2D motions) how it is sometime useful to work with respect to a moving coordinate system. In particular the question of how one can go from a static coordinate system to a moving coordinate system. For example, consider a passenger in a train. One can talk about the passenger’s position with respect to the ground or with respect to the train. Clearly the two things are not the same.

More generally, let object \(A\) move with respect to a coordinate system (or reference frame) of object \(B\), described by position vector

\[
\vec{r}_{A/B}(t)
\]\[3.60\]

and let object \(B\) move with respect to a coordinate system (or reference frame) of object \(C\), described by position vector

\[
\vec{r}_{B/C}(t)
\]\[3.61\]

then we say that to describe object \(A\) with respect to a coordinate system (or reference frame) of object \(C\), we use the following rule

\[
\vec{r}_{A/C}(t) = \vec{r}_{A/B}(t) + \vec{r}_{B/C}(t).
\]\[3.62\]

(Note that this rule works very well only for small velocities, but fails miserably when velocities approach the speed of light which is the realm of the so-called special theory of relativity.)

By taking a time derivative of Eq. (3.62) we get a rule for adding velocities

\[
\frac{d\vec{r}_{A/C}(t)}{dt} = \frac{d\vec{r}_{A/B}(t)}{dt} + \frac{d\vec{r}_{B/C}(t)}{dt}
\]

\[
\vec{v}_{A/C}(t) = \vec{v}_{A/B}(t) + \vec{v}_{B/C}(t).
\]\[3.63\]
Example 3.14-15. There is a 100 km/h wind from west to east. (a) If an airplane’s compass indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 240 km/h, what is the velocity of the airplane relative to earth? (b) What direction should the pilot head with speed 240 km/h to travel due north?

Step 1: Choose coordinate system. Let x-axis to point east and y-axis to point north.

Step 2: What is given?

\[ \vec{v}_{A/E} = 100 \text{ km/h} \hat{i} \]

Step 3: What do we have to find? (a) The airspeed indicator tells us that

\[ \vec{v}_{P/A} = 240 \text{ km/h} \hat{j} \]

and by using (3.63) we get

\[ \vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E} = 100 \text{ km/h} \hat{i} + 240 \text{ km/h} \hat{j} \]

(b) In general velocity of airplane relative to air is

\[ \vec{v}_{P/A} = x \hat{i} + y \hat{j} \]

and (3.63) we get

\[ \vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E} = \left( x \hat{i} + y \hat{j} \right) + 100 \text{ km/h} \hat{i} = (x + 100 \text{ km/h}) \hat{i} + y \hat{j}. \]

where

\[ x + 100 \text{ km/h} = 0 \]

But since

\[ v_{P/A} = \sqrt{x^2 + y^2} = 240 \text{ km/h} \]

we have two equations with two unknowns with solution

\[ x = -100 \text{ km/h} \]
\[ y = 218 \text{ km/h}. \]

and thus

\[ \vec{v}_{P/A} = -100 \text{ km/h} \hat{i} + 218 \text{ km/h} \hat{j}. \]