Chapter 6

Work and Kinetic Energy

6.1 Work

We have already seen that (according to Newton’s First Law) velocity of an object is unchanged if the sum of all forces is acting on it is zero. We have also seen that the speed of an object is unchanged if the sum of all forces is acting in the direction orthogonal to the direction of motion. An important observation is that in both cases (sum of forces is zero or sum of forces is orthogonal to direction of motion) the speed of an object remains constant and thus any function of speed (i.e. magnitude of velocity) would not change, e.g.

\[ f(v(t)) \propto v(t)^2 = \text{const.} \quad (6.1) \]

When does the speed change? For that a force acting on the body must have a component parallel to the direction of displacement. In the case of a straight-line displacement, \( \vec{s} \) and a constant force \( \vec{F} \), the relevant quantity can be written using dot product, i.e.

\[ W = \vec{F} \cdot \vec{s} = Fs \cos \phi. \quad (6.2) \]

This new physical quantity is known as work. Although the work is done by each and every force acting on a body, according to above equation it may be positive, negative or even zero.

The units of work are given by

\[ [\text{Work}] = [\text{Force}] \times [\text{Distance}]. \quad (6.3) \]

For example in SI units we define one unit of joule as a work done by a force of one newton to move an object on distance one meter, i.e.

\[ 1 \text{ J} = 1 \text{ N} \times 1 \text{ m.} \quad (6.4) \]
Note that the dimensions of work can be also written as

\[ \text{[Work]} = [\text{Mass}] \times [\text{Acceleration}] \times [\text{Distance}] \]  \hspace{1cm} (6.5)

or

\[ \text{[Work]} = [\text{Mass}] \times [\text{Velocity}]^2. \]  \hspace{1cm} (6.6)

This will be important when we discuss conservation of energy in the following section.

**Example 6.2.** A farmer hitches her tractor to a sled loaded with firewood and pulls it a distance of 20 m along ground. The total weight of sled and load is 147000 N. The tractor exerts a constant 5000 N force at an angle of 36.9° above the horizontal. A 3500 N friction force opposes the sled’s motion. Find the work done by each force acting on the sled and the total work done by the forces.

Step 1: Coordinate system. Let’s choose a coordinate system with x-axis pointing horizontally in the direction of motion of the sled, and y-axis pointing vertically.

Step 2: Draw a free body diagram.

Step 3: Apply Newton’s laws if needed to calculate all forces

\[
\vec{w} = -147000 \text{N}\hat{j} \\
\vec{n} = n\hat{j} \\
\vec{F}_T = 5000 \text{N} \cos(36.9°) \hat{i} + 5000 \text{N} \cos(36.9°) \hat{i} \\
\vec{f} = -3500 \text{N} \hat{i}
\]  \hspace{1cm} (6.7)
and then calculate work done by each force

\[ W_w = (-147000 \text{N} \hat{j}) \cdot (20 \text{m} \hat{i}) = 0 \]
\[ W_n = (n \hat{j}) \cdot (20 \text{m} \hat{i}) = 0 \]
\[ W_T = (5000 \text{N} \cos (36.9^\circ) \hat{i} + 5000 \text{N} \cos (36.9^\circ) \hat{i}) \cdot (20 \text{m} \hat{i}) = 5000 \text{N} \cos (36.9^\circ) 20 \text{m} = 80 \text{kJ} \]
\[ W_f = (-3500 \text{N} \hat{i}) \cdot (20 \text{m} \hat{i}) = -70 \text{kJ}. \]

### 6.2 Kinetic Energy

Back in second chapter we argued that 1D motion with constant acceleration is described by following equations

\[
\begin{align*}
  x(t) &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \\
  v(t) &= v_{0x} + a_x t \\
  a(t) &= a_x. \quad (6.9)
\end{align*}
\]

Then we derived a useful relation by expressing \( t \) in the equation for velocity

\[
t = \frac{v_x(t) - v_{0x}}{a_x} \quad (6.10)
\]

and by plugging it in the equation for position

\[
\begin{align*}
  x(t) &= x_0 + v_{0x} \left( \frac{v_x(t) - v_{0x}}{a_x} \right) + \frac{1}{2} a_x \left( \frac{v_x(t) - v_{0x}}{a_x} \right)^2 \\
  x(t) - x_0 &= \frac{v_x(t) \cdot v_{0x} - v_{0x}^2}{a_x} + \frac{v_x(t)^2 - 2 v_x(t) v_{0x} + v_{0x}^2}{2 a_x} \\
  2a_x (x(t) - x_0) &= 2v_x(t) \cdot v_{0x} - 2v_{0x}^2 + v_x(t)^2 - 2 v_x(t) v_{0x} + v_{0x}^2 \\
  2a_x (x(t) - x_0) &= v_x(t)^2 - v_{0x}^2 \quad (6.11)
\end{align*}
\]

which can be written as

\[
ma_x (x - x_0) = \frac{1}{2} mv_x^2 - \frac{1}{2} mv_{0x}^2. \quad (6.12)
\]

Note that this equation does not depend on time explicitly, but only through time-dependence of \( x \) and \( v_x \).

In 3D the equation can be written as

\[
m \vec{a} \cdot \vec{s} = \frac{1}{2} m v^2 - \frac{1}{2} m v_0^2 \quad (6.13)
\]
where the left hand side takes the form of the equation for work 6.2 with following identifications

\[
\vec{F} = m\vec{a}
\]
\[
\vec{s} = (x - x_0, y - y_0, z - z_0).
\] (6.14)

How should interpret the right hand side of (6.13)? If we denote by

\[
K(v) = \frac{1}{2}mv^2
\] (6.15)

another physical quantity (known as kinetic energy), then equation (6.13) tells us that the work acting on a system changes its kinetic energy by the amount equal to work

\[
W = K(t) - K(t_0) = \Delta K.
\] (6.16)

In fact what is relevant is the total work of all forces done on a system, i.e.

\[
W_{\text{tot}} = \Delta K.
\] (6.17)

This is called the work-energy theorem or an equation representing conservation of energy.

**Example 6.3.** Let’s come back to Example 6.2. Suppose sled’s initial speed \(v_1 = 2.0 \text{ m/s}\).
What is the speed of the sled after it moves 20 m?

Step 2: Initial Conditions. The initial velocity is

\[ v_1 = 2.0 \text{ m/s} \hat{i} \]

and thus the initial kinetic energy is

\[ K_1 = \frac{1}{2} (1500 \text{ kg}) (2.0 \text{ m/s})^2 = 3000 \text{ J}. \]

Step 3a: Apply the work-energy theorem. From Eqs. (6.8) the total work is

\[ W_{\text{tot}} = W_w + W_n + W_T + W_f = 10 \text{ kJ} \quad (6.18) \]

and mass of the sled is

\[ m = \frac{147000 \text{ N}}{9.8 \text{ m/s}^2} = 1500 \text{ kg}. \quad (6.19) \]

Therefore according to conservation of energy (6.17) all of it work had to be used to change kinetic energy

\[ W_{\text{tot}} = K_2 - K_1 \]

\[ 10 \text{ kJ} = K_2 - 3000 \text{ J}. \quad (6.20) \]

to its final value

\[ K_2 = 13000 \text{ J}. \quad (6.21) \]

then according to (6.15) the final speed is

\[ v_2 = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2 (13000 \text{ N})}{1500 \text{ kg}}} = 4.2 \text{ m/s}. \quad (6.22) \]

Step 3b: Determine the final conditions. Since the motion was with constant force according to Newton’s second law the constant acceleration is

\[ \vec{a} = \left( \frac{5000 \text{ N} \cos (36.9^\circ) \hat{i} - 3500 \text{ N} \hat{i}}{1500 \text{ kg}} \right) = 0.333 \text{ m/s}^2 \hat{i} \quad (6.23) \]

we can use (6.11)

\[ v_2^2 = (2.0 \text{ m/s})^2 + 2 (0.333 \text{ m/s}^2) (20 \text{ m}) = 17.3 \text{ m}^2/\text{s}^2 \quad (6.24) \]

to get

\[ v_2 = 4.2 \text{ m/s} \quad (6.25) \]
which clearly agrees with (6.22).

**Example 6.4.** The 200 kg steel hammerhead of a pile driver is lifted 3.00 m above the top of a vertical I-beam being driven into the ground. The hammerhead is then dropped, driving the I-beam 7.4 cm deeper into the ground. The vertical guide rails exert a constant 60 N friction force on the hammerhead. Use the work-energy theorem to find (a) the speed of the hammerhead just as it heats the I-beam and (b) the average force the hammerhead exerts on the I-beam. Ignore effects of air.

Step 1: Coordinate system. Let’s chose a 1D coordinate system with x-axis pointing upward.

Step 2: Draw a free-body diagram.

Step 3: Apply the work-energy theorem. The total force acting on the hammerhead in a free fall is

\[ \mathbf{F}_{1-2} = - (200 \text{ kg}) (9.8 \text{ m/s}^2) \mathbf{j} + 60 \mathbf{j} = -1900 \mathbf{j} \]

and as it falls down the work is

\[ W_{1-2} = \mathbf{F}_{1-2} \cdot \mathbf{s} = \left( -1900 \mathbf{j} \right) \cdot \left( -3.00 \text{ m} \mathbf{j} \right) = 5700 \text{ J.} \]

The total force acting on the hammerhead when it is in contact with I-beam

\[ \mathbf{F}_{2-3} = - (200 \text{ kg}) (9.8 \text{ m/s}^2) \mathbf{j} + 60 \mathbf{j} + n \mathbf{j} \]
and as it moves through the ground the work is

\[ W_{2-3} = \left(- (200 \text{ kg}) (9.8 \text{ m/s}^2) \hat{j} + 60 \hat{n} + n \hat{j} \right) \cdot \left(-0.074 \text{ m} \hat{j} \right) = 140 \text{ J} - n \cdot 0.074 \text{ m}. \]  

(6.29)

The work-energy theorem tells us that

\[ W_{1-2} = K_2 - K_1 \]  

(6.30)

and

\[ W_{2-3} = K_3 - K_2. \]  

(6.31)

But since \( v_1 = v_3 = 0 \) (hammer does not move before it is dropped and the I-beam does not move at the end) we get the following equations

\[
\begin{align*}
W_{1-2} &= K_2 \\
W_{2-3} &= -K_2 
\end{align*}
\]  

(6.32)

which implies that

\[ \sqrt{\frac{2W_{1-2}}{m}} = \sqrt{\frac{2 \cdot 5700 \text{ J}}{200 \text{ kg}}} = 7.55 \text{ m/s} \]  

(6.33)

and

\[
\begin{align*}
W_{1-2} &= -W_{2-3} \\
140 \text{ J} - n \cdot 0.074 \text{ m} &= -5700 \text{ J} \\
n &= \frac{5840 \text{ J}}{0.074 \text{ m}} = 79000 \text{ N}. 
\end{align*}
\]  

(6.34)

Then (from Newton’s third law) the force acting on the I-beam is

\[ \vec{F}_I = -n = -79000 \text{ N} \hat{j}. \]  

(6.35)

Example 6.5. Two iceboats hold a race on a frictionless horizontal lake.
The two iceboats have masses \( m \) and \( 2m \). The iceboats have identical sails, so the wind exerts the same constant force \( \vec{F} \) on each iceboat. They start from rest and cross the finish line a distance \( s \) away. Which iceboat crosses the finish line with greater kinetic energy.

Step 1: Coordinate system. Let’s choose a 1D coordinate system with \( x \)-axis pointing to the right.

Step 2: Draw a free body diagram. Along \( x \)-axis there is only one force pushing the iceboats to the right, i.e.

\[
\vec{F} = F\hat{i}. \tag{6.36}
\]

for each boat.

Step 3: Apply Newton’s Laws / Work-energy theorem. The latter implies

\[
W = K_f - K_i. \tag{6.37}
\]

Both boats start from rest and so have zero kinetic energy

\[
K_i = 0 \tag{6.38}
\]

but at after being displaced by vector

\[
\vec{s} = s\hat{i} \tag{6.39}
\]

the kinetic energy of each boat is

\[
K_f = W - K_i = \left( F\hat{i} \right) \left( s\hat{i} \right) = Fs. \tag{6.40}
\]
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On the other hand the velocities of boats are different

\[ \frac{1}{2}mv_1^2 = Fs \Rightarrow \vec{v}_1 = \sqrt{\frac{2Fs}{m}} \hat{i} \]

\[ \frac{1}{2}2mv_2^2 = Fs \Rightarrow \vec{v}_2 = \sqrt{\frac{Fs}{m}} \hat{i}. \]  

(6.41)

6.3 Varying Forces

Up to now we were dealing with work done by constant forces when the object is moving along a straight line. If the direction of motion is along x-axis then the work done by force \( \vec{F} \) is given by

\[ W = F_x (x_f - x_i). \]  

(6.42)

We can generalize this set-up in two different ways. First of all we can consider a motion along a curve and/or we can consider forces which change with position.

In the latter case the motion can be broken into sufficiently small segments such that the force along each segment does not vary much. Then we can still apply (6.42) to each segment separately, i.e

\[ W_1 = F_1 \Delta x \]
\[ W_2 = F_2 \Delta x \]
\[ \vdots \]
\[ W_n = F_n \Delta x \]  

(6.43)

and by adding all these small pieces of work we get

\[ \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} F_i \Delta x \]  

(6.44)

in the limit of infinite \( n \) this sum reduces to integral

\[ W \equiv \lim_{n \to \infty} \sum_{i=1}^{n} F_i \Delta x = \int_{x_i}^{x_f} F_x(x) dx \]  

(6.45)

where \( F_x(x) \) is the x-component of force \( \vec{F}(x) \) which is a function of position. (Note that in the case when \( F_x(x) \) is a constant force (6.45) reduces to (6.42).)

For example, force required to stretch an ideal spring is proportional to the spring’s elongation

\[ F_x(x) = kx. \]  

(6.46)
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where $k$ is the so-called spring constant measured

$$[k] = \frac{[\text{Force}]}{[\text{Distance}]} = \frac{[\text{Mass}]}{[\text{Time}]^2}.$$  

From (6.45) and (6.46) the work need to stretch an ideal spring is

$$W = \int_0^X F_x(x)dx$$

$$= \left[ \frac{1}{2}kx^2 \right]_0^X$$

$$= \frac{1}{2}kX^2.$$  \hspace{1em} (6.47)

Note that the non-stretched spring in (6.46) and (6.47) corresponds to $x = 0$, but if the origin is displaced then the equation would have to be modified

$$F_x(x) = k(x - x_0)$$

$$W = \frac{1}{2}k(X - x_0)^2.$$  \hspace{1em} (6.48)

Although the force can be either negative or positive, work needed to stretch an ideal spring from a non-stretched state must be positive, but if in the initial state the spring was already stretched, then

$$W = \int_{x_1}^{x_2} F_x(x)dx$$

$$= \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2.$$  \hspace{1em} (6.49)

can be either negative, positive or zero.

Example 6.6. A woman weighing 600 N steps on a bathroom scale that contains a stiff spring. In equilibrium, the spring is compressed 1.0 cm under
her weight. Find the force constant of the spring and the total work done on it during compression.

Step 1: Coordinate system. Choose a coordinate system with x-axis pointing upwards.

Step 2: Free body diagram. There are two forces acting on the scale one upward due to gravity and one downward due to spring tension.

Step 3: Apply Newton’s Laws / Work-energy theorem. From Third Law the force of woman acting on the spring and force of spring acting on woman are related

\[ \vec{F}_{\text{on s}} = -\vec{F}_{\text{on w}} \]  \hspace{1cm} (6.50)

and from First Law

\[ \vec{W} + \vec{F}_{\text{on w}} = 0 \]  \hspace{1cm} (6.51)

and thus

\[ \vec{F}_{\text{on s}} = -600 \hat{i}. \]  \hspace{1cm} (6.52)

Since the spring had compressed by

\[ \vec{s} = -1.0 \text{ cm} \hat{i} \]  \hspace{1cm} (6.53)

the spring constant is

\[ k = \frac{-600 \text{ N}}{-0.010 \text{ m}} = 6.0 \times 10^4 \text{ N/m}. \]  \hspace{1cm} (6.54)

and work done on the spring

\[ W = \frac{1}{2} \left( 6.0 \times 10^4 \text{ N/m} \right) (-0.010 \text{ m})^2 = 3.0 \text{ J} \]  \hspace{1cm} (6.55)
Example 6.7. An air-track glider of mass 0.100 kg is attached to the end of a horizontal air track by a spring with force constant 20.0 N/m. Initially the spring is unstretched and the glider is moving at 1.50 m/s to the right. Find the maximum distance \( d \) that the glider moves to the right (a) is the air track is turned on, so that there is no friction, and (b) if the air is turned off, so that there is a kinetic friction coefficient \( \mu_k = 0.47 \).  

Step 1: Coordinate system. Let’s choose x-axis to point to the right with origin at the location of mass when spring is unstretched.  
Step 2 (a): Draw a free body diagram.  

Step 2 (b): Initial Conditions. The initial velocity is \( \vec{v}_i = 1.50 \, \text{m/s} \, \hat{i} \) and thus the initial kinetic energy is  
\[
K_i = \frac{1}{2}mv_i^2 = \frac{1}{2}(0.100 \, \text{kg})(1.50 \, \text{m/s})^2 = 0.113 \, \text{J}
\]  

Step 3: Apply Newton Laws / Work-Energy theorem. With or without friction, when the glider moves the maximum distance to the right the velocity is \( \vec{v}_f = 0 \, \text{m/s} \, \hat{i} \) and thus the initial kinetic energy is  
\[
K_f = 0 \, \text{J}.
\]  

(a) In the case of no friction there is only one force which has a component along x-axis  
\[
\vec{F}_s = -F_s \hat{i} = -k \, x \, \hat{i}
\]
and so the work done by this force is

\[
W_s = -\frac{1}{2} (20.0 \text{ N/m}) x^2
\]  

(6.61)

Applying the work-energy theorem gives us

\[
W_s = K_f - K_i
\]

(6.62)

\[
-\frac{1}{2} (20.0 \text{ N/m}) x^2 = 0 \text{ J} - 0.113 \text{ J}
\]

and (by only choosing the physically-relevant positive solution) we get

\[
x = \sqrt{\frac{0.113 \text{ J}}{10.0 \text{ N/m}}} = 0.106 \text{ m.}
\]  

(6.63)

(a) In the case with friction in addition to the work done by spring

\[
W_s = -\frac{1}{2} (20.0 \text{ N/m}) x^2
\]  

(6.64)

there is also work done by frictional force

\[
\vec{f}_k = -\mu_k mg \hat{i} = - (0.47) (0.100 \text{ kg} \times 9.8 \text{ m/s}^2) \hat{i} = -0.46 \text{ N} \hat{i}
\]  

(6.65)

which is

\[
W_f = \left( -0.46 \text{ N} \hat{i} \right) \left( x \hat{i} \right) = -0.46 \text{ N} x.
\]  

(6.66)

Applying the work-energy theorem gives us a quadratic equation

\[
W_s + W_f = K_f - K_i
\]

\[
(-10.0 \text{ N/m}) x^2 - (0.46 \text{ N}) x = -0.113 \text{ J}
\]  

(6.67)

with solutions

\[
ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
(-10.0) x^2 - (0.46) x + 0.113 = 0 \implies x = \frac{(0.46) \pm \sqrt{(0.46)^2 + 4 (10.0) (0.113)}}{2 (-10.0)}
\]  

(6.68)

but only one (physically-relevant) positive solution

\[
x = 0.086 \text{ m.}
\]  

(6.69)
6.4 Power

So far we did not pay attention to how much time it takes to do the work. The physical quantity which represents the rate at which the work is done is called power. This suggests that dimensions of power should be

\[
[\text{Power}] = \frac{[\text{Work}]}{[\text{Time}]}.
\]

(6.70)

In SI units power is measured in watts (W) which is defined as

\[
1 \text{W} = \frac{1 \text{J}}{1 \text{s}}
\]

(6.71)

The average power is defined from total work done in some interval in exactly the same way as average velocity was defined from displacement during some interval,

\[
\begin{align*}
\bar{v} &= \frac{\Delta x}{\Delta t} \\
\bar{P} &= \frac{\Delta W}{\Delta t}.
\end{align*}
\]

(6.72)

Similarly the (if you wish instantaneous) power is defined as instantaneous velocity by taking limit of interval to zero

\[
\begin{align*}
v &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \\
P &= \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}.
\end{align*}
\]

(6.73)

Since work is given by

\[
W = \vec{F} \cdot \vec{s}
\]

(6.74)

the power can be expressed as

\[
P = \frac{dW}{dt} = \frac{d}{dt} \left( \vec{F} \cdot \vec{s} \right) = \frac{d\vec{F}}{dt} \cdot \vec{s} + \vec{F} \cdot \frac{d\vec{s}}{dt}
\]

(6.75)

but for constant (time-independent) forces acting on a particle

\[
\frac{d\vec{F}}{dt} = 0
\]

(6.76)

the power is

\[
P = \vec{F} \cdot \vec{v}
\]

(6.77)
where
\[ \vec{v} = \frac{d\vec{s}}{dt}. \] (6.78)
is the velocity of particle.

**Example 6.9.** Each of the four jet engines on an Airbus A380 airliner develops a thrust (a forward force on the airliner) of 322000 N (72000 lb). When the airplane is flying at 250 m/s (900 km/h, or roughly 560 mi/h) what is horsepower does each engine develop.

Step 1: Coordinate system. x-axis can be chosen to point along the direction of motion of the airplane such that
\[ \vec{v} = 250 \text{ m/s} \hat{i}. \] (6.79)

Step 2: Free body diagram. There are few forces acting on the airplane, but only thrust forces have a non-zero component along x-axis
\[ \vec{F}_t = 322000 \text{ N} \hat{i}. \] (6.80)

Step 3: Calculate power. According to (6.77) the powers is given by
\[ P = 322000 \text{ N} \hat{i} \cdot 250 \text{ m/s} \hat{i} = 8.05 \times 10^7 \text{ W} \] (6.81)
or
\[ P = 8.05 \times 10^7 \text{ W} \frac{1 \text{ hp}}{746 \text{ W}} = 108,000 \text{ hp}. \] (6.82)