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On sequential estimation for branching processes with immigration

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Abstract

Consider a Galton–Watson process with immigration. The limiting distributions of the non-sequential estimators of the offspring mean have been proved to be drastically different for the critical case and subcritical and supercritical cases. A sequential estimator, proposed by Sriram et al. (Ann. Statist. 19 (1991) 2232), was shown to be asymptotically normal for both the subcritical and critical cases. Based on a certain stopping rule, we construct a class of two-stage estimators for the offspring mean. These estimators are shown to be asymptotically normal for all the three cases. This gives, without assuming any prior knowledge, a unified estimation and inference procedure for the offspring mean.

Keywords: Two-stage sequential estimator; Stopping time; Asymptotic normality; Branching process

1. Introduction

Since the pioneering work by Heyde and Seneta (1972, 1974), the study of the estimation problem for Galton–Watson processes with immigration has drawn much attention. Early work on the estimates of means of offspring and immigration dates back to Smoluchowski (1916). Later work considered under some parametric models are given in Bhat and Adke (1981), Venkataraman (1982) and Venkataraman and Nanthi (1982).

Let m denote the mean of the offspring population. The three cases (subcritical, critical or supercritical) are distinguished according as $m < 1$, $m = 1$ or $m > 1$.

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Various estimates of the parameter m can be found in Heyde and Seneta (1971), Heyde (1970), Quine (1976), Klimko and Nelson (1978) and Wei and Winnicki (1987) under different cases.

Wei and Winnicki (1990) proposed a unified estimator by using a conditional weighted least-squares method.

However, the limit distributions for all these estimators of m in the critical case, drastically different from that of the other two cases, are nonnormal. This raises the question of how to make inference on m if we have no prior knowledge on m.

Based on full information on both generation sizes and the immigration process, Sriram et al. (1991) proposed a sequential estimator for $m \in (0, 1]$. It is shown that this sequential estimator is consistent and asymptotically normal. Later, Shete and Sriram (1998) modified the sequential estimator and constructed a so-called fixed-precision estimator. The modified estimator is proved to be unbiased for all $m \in (0, \infty)$, as well as asymptotically efficient (the same as the sequential estimator) in case $m \in (0, 1]$. The asymptotic properties for both the sequential estimator and the modified estimator are still unknown for the supercritical case.

The motivation for the sequential estimator, however, comes from the work by Lai and Siegmund (1983) for autoregressive (AR) processes of order 1. Lai and Siegmund (1983) proposed a stopping time for AR(1) processes in the context of estimating the autoregressive parameter β and established the uniform asymptotic normality of the stopping least-squares estimator in case $|\beta| \leqslant 1$. Shiryaev and Spokoiny (1997) proved the asymptotic normality of the sequential least-squares estimator in the case $|\beta| > 1$ by assuming the normal error with known variance.

This suggests that there may be a unified approach available for finding the limit distribution of the sequential estimator for the offspring mean. Simulation studies in Shete and Sriram (1998) also give a hint that the sequential estimator might be asymptotically normal in the supercritical case, while the modified estimator might be asymptotically a scale mixture of normals. See also Shete (1998).

This paper focuses on the study of sequential estimation for the mean of the offspring. Based on the stopping rule of Sriram et al. (1991), we construct two-stage estimators which are strongly consistent and asymptotically normal for all $m > 0$. Section 2 introduces our two-stage sequential estimators and states the main results of the paper; Section 3 provides the proofs. Finally, Section 4 demonstrates some simulation results.

2. Sequential estimation

The branching process with immigration is defined by

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n-1,k} + Y_n, \quad n \geqslant 1, \tag{2.1}$$

where Z_n is the size of the nth generation of a population, $\xi_{n-1,k}$ is the number of offspring of the kth individual belonging to the $(n-1)$th generation and Y_n denotes the number of immigrants in the nth generation.

Suppose that $\{\xi_{n-1,k},\ n \geqslant 1,\ k \geqslant 1\}$ and $\{Y_n,\ n \geqslant 1\}$ are independent array and sequence of independent and identically distributed (i.i.d.) integer-valued random variables. The initial size Z_0 is a random variable independent of $\{\xi_{n,j}\}$ and $\{Y_n\}$. The offspring and the immigration are assumed to have unknown means m and λ, and variances $\sigma^2 \in (0,\infty)$ and $\sigma_Y^2 \in (0,\infty)$, respectively.

Throughout we assume that both $\{Z_n\}$ and $\{Y_n\}$ are observable. A natural estimator for the offspring mean is given by

$$\hat{m}_n = \sum_{i=1}^{n} (Z_i - Y_i) \Big/ \sum_{i=1}^{n} Z_{i-1}. \tag{2.2}$$

The estimator can be shown to be an MLE of m if one assumes the power-series offspring and immigration distribution (a power-series distribution F has the form $F(u) = a(u)\theta^u/A(u)$, for $u \geqslant 0$, where $a(u) \geqslant 0$ and $A(\theta) = \sum_{u=0}^{\infty} a(u)\theta^u$ for $0 < \theta < \theta^*$).

Instead of using a fixed number of observations, Sriram et al. (1991) defined the stopping rule by

$$N_c = \inf\left\{ n \geqslant 1\colon \sum_{i=1}^{n} Z_{i-1} \geqslant c\sigma^2 \right\}, \tag{2.3}$$

where $c > 0$ is chosen appropriately. The sequential estimator of m is then given by $\hat{m}_{N_c}$. The σ^2 is involved in the definition of N_c in order to prove the uniformly asymptotic normality and to control the expected mean squared error of estimation. The N_c is well defined if σ^2 is known.

For simplicity of proofs, in this paper we assume σ^2 is known in definition (2.3). When σ^2 is unknown, one can absorb the σ^2 into the constant c and instead define

$$N(c) = \inf\left\{ n \geqslant 1\colon \sum_{i=1}^{n} Z_{i-1} \geqslant c \right\} \tag{2.4}$$

and then use $\hat{m}_{N(c)}$ as a sequential estimator of m.

Now let $G = G(c, N_c) \geqslant N_c$ be an integer-valued function of c and N_c satisfying that as $c \to \infty$,

$$\frac{G(c,N_c) - N_c}{c^{\alpha}} \to 0 \quad \text{a.s. for some } \alpha \in (0, \tfrac{1}{2}) \text{ if } m \in (0,1] \tag{2.5}$$

or

$$G(c,N_c) - N_c \to \infty \quad \text{a.s. if } m > 1. \tag{2.6}$$

Our two-stage sequential estimator is defined by $\hat{m}_G$. The main result of the paper is the following theorem.

Theorem 2.1. *$\hat{m}_G$ converges almost surely to m as c tends to infinity. If $G = G(c, N_c)$ satisfies* (2.5) *and* (2.6), *then*

$$\left(\sum_{i=1}^{G} Z_{i-1} \right)^{1/2} \frac{\hat{m}_G - m}{\sigma} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1). \tag{2.7}$$

Note 1. It is easy to show that conditions (2.5) and (2.6) can be satisfied for a wide class of functions. To this end, let $g(c)$ be any integer-valued function satisfying

$$\lim_{c\to\infty} g(c) = \infty \quad \text{and} \quad \lim_{c\to\infty} \frac{g(c)}{c^{\alpha}} = 0 \text{ for some } \alpha \in (0, \tfrac{1}{2}). \tag{2.8}$$

If we set $G = G(c, N_c) = g(c) + N_c$, then (2.5) and (2.6) hold automatically.

We may also define $G(c) = G(c, N_c) = \max(g(c), N_c)$, i.e.

$$G(c) = \inf \left\{ n \geqslant g(c) \colon \sum_{i=1}^{n} Z_{i-1} \geqslant c\sigma^2 \right\},$$

where $g(c)$ satisfies

$$\lim_{c\to\infty} \frac{g(c)}{\log c} = \infty$$

in addition to (2.8). To prove (2.5) and (2.6), we need the following lemma.

Lemma 2.2. *Let N_c be defined as in* (2.3).

(i) *If $m \in (0,1)$, then*

$$\lim_{c\to\infty} \frac{N_c}{c} = \frac{(1-m)\sigma^2}{\lambda} \quad \textit{a.s.}$$

(ii) *if $m = 1$, then*

$$\liminf_{c\to\infty} \frac{N_c}{\sqrt{c/\log c}} = \infty \quad \textit{and} \quad \limsup_{c\to\infty} \frac{N_c}{c} < \infty \quad \textit{a.s.}$$

(iii) *if $m > 1$, then*

$$\lim_{c\to\infty} \frac{N_c}{\log c} = \frac{1}{\log m} \quad \textit{a.s.}$$

It is obvious from Lemma 2.2 that if $m \leqslant 1$

$$\lim_{c\to\infty} \frac{N_c}{g(c)} = \infty \quad \text{a.s.}$$

and thus $G = N_c$ a.s. eventually, and if $m > 1$ then

$$\lim_{c\to\infty} \frac{N_c}{g(c)} = 0 \quad \text{a.s.}$$

and $G = g(c)$ a.s. eventually. Therefore, both (2.5) and (2.6) are trivial.

Note 2. Theorem 2.1 remains true if one replaces N_c by $N(c)$. And if σ^2 is unknown, one must proceed in this way. But to give an asymptotic confidence interval for m we still need an estimator for σ^2. Shete and Sriram (1998) defined the following estimator:

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{(Z_i - \hat{m}_n Z_{i-1} - Y_i)^2}{Z_{i-1}}. \tag{2.9}$$

They proved that $\hat{\sigma}_n^2 \to \sigma^2$ a.s. as $n \to \infty$ provided that $E\xi_{1,1}^4 < \infty$. Since $G \to \infty$ a.s., it is trivial that $\hat{\sigma}_G^2 \to \sigma^2$ a.s. as $c \to \infty$. An immediate consequence is obtained as follows.

Corollary 2.3. *Assume that* $G = G(c, N(c))$, *where* $N(c)$ *is defined as in* (2.4), *satisfies conditions* (2.5) and (2.6) *when* N_c *is replaced by* $N(c)$, *and that* $E\xi_{1,1}^4 < \infty$. *Then*

$$\left(\sum_{i=1}^{G} Z_{i-1}\right)^{1/2} \frac{\hat{m}_G - m}{\hat{\sigma}_G} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1).$$

3. Proofs

Let $\{\xi_i\}$ be i.i.d. random variables distributed the same as $\xi_{1,1}$ and independent of Z_0 and $\{Y_n\}$.

For any nonnegative integer t define

$$S_1(t) = Z_0'(t) = t \quad \text{and} \quad Z_1'(t) = \sum_{i=1}^{Z_0'(t)} \xi_i + Y_1.$$

Recursively, for $n \geqslant 2$ define

$$S_n(t) = \sum_{i=1}^{n} Z_{i-1}'(t) \quad \text{and} \quad Z_n'(t) = \sum_{i=S_{n-1}(t)+1}^{S_n(t)} \xi_i + Y_n.$$

Then, it is easy to show that the sequences $\{Z_{n-1}, Y_n,\ n \geqslant 1\}$ and $\{Z_{n-1}'(Z_0), Y_n,\ n \geqslant 1\}$ have the same joint distribution. This can be done by showing that $\{Z_{n-1}\}$ and $\{Z_{n-1}'\}$, given $\{Y_n,\ n \geqslant 1\}$, have the same joint distribution. Thus, without loss of generality, we can simply assume

$$\hat{m}_n = \sum_{i=1}^{n} (Z_i' - Y_i) \Bigg/ \sum_{i=1}^{n} Z_{i-1}'$$

and

$$N_c = \inf\{n \geqslant 1\colon S_n \geqslant c\sigma^2\},$$

where Z_i' and S_n denote for $Z_i'(Z_0)$ and $S_n(Z_0)$, respectively when the initial population size is Z_0.

Obviously,

$$\hat{m}_n - m = \frac{1}{S_n} \sum_{i=1}^{S_n} (\xi_i - m). \tag{3.1}$$

Note that N_c is well defined. Since $S_n \geqslant \sum_{i=1}^{n-1} Y_i$, we have $S_n \to \infty$ a.s. by the strong law of large numbers. Thus, $P(N_c < \infty) = 1$ and $N_c \to \infty$ a.s. as $c \to \infty$.

Lemma 3.1. *Assume* $m \in (0, 1]$ *then* $Z_n' = \mathrm{o}(n \log n)$ *a.s.*

Proof. The proof follows from that of Lemma 2.1 in Wei (1991). □

Lemma 3.2. *For* $c > 0$, $\{\xi_{N_c+j},\ j \geqslant 1\}$ *and* (S_{N_c}, Z'_{N_c}, N_c) *are independent, and moreover,* $\{\xi_{N_c+j},\ j \geqslant 1\}$ *is a sequence of i.i.d. random variables distributed as* ξ_1.

The proof of the lemma is simple and is omitted.

Since Lemma 2.2 will be used in the proof of Theorem 2.1, we will prove it first.

Proof of Lemma 2.2. (i) See Sriram et al. (1991) for $m \in (0,1)$.

(ii) Let $m = 1$.

Note that $S_n \geqslant \sum_{i=1}^{n-1} Y_i$. By the strong law of large numbers, $\liminf_{n\to\infty} S_n/n \geqslant \lambda$ a.s. and hence $\liminf_{c\to\infty} S_{N_c-1}/(N_c - 1) \geqslant \lambda$. Therefore, $\limsup_{c\to\infty} N_c/c < \infty$ a.s. since $S_{N_c-1} < c\sigma^2$.

From Lemma 3.1 we have $S_n = \sum_{i=1}^{n} Z_{i-1} = \mathrm{o}(n^2 \log n)$ a.s., which yields

$$\frac{N_c^2 \log N_c}{c} \to \infty \quad \text{a.s.}$$

Hence

$$\frac{N_c(\log c)^{1/2}}{c^{1/2}} \to \infty \quad \text{a.s.}$$

(iii) For the supercritical case $m > 1$, let $W_n = Z'_n/m^n$. Seneta (1970) showed that

$$W_n \to W \quad \text{a.s., where } W \text{ is a positive random variable.}$$

So we can conclude that

$$\frac{S_n}{m^n} \to \frac{W}{m-1} \quad \text{a.s.} \tag{3.2}$$

which simply implies part (iii) of the lemma. □

The following lemma is a generalization of Theorem 17.1 of Billingsley (1968, p. 146).

Lemma 3.3. *Assume that* $\{\eta_j,\ j \geqslant 1\}$ *is a sequence of i.i.d. random variables with* $E\eta_1 = 0$ *and* $E\eta_1^2 = 1$ *and* $\{H_n\}$ *is a sequence of random vectors. For each* $n \geqslant 1$, $\{\eta_j\}$ *and* H_n *are independent. If* $\{v_n\}$ *and* $\{u_n\}$ *are two sequences of integer-valued random variables such that* v_n *is measurable with respect to* $\sigma(H_n)$ *and* u_n *is the function of* $\{\eta_j\}$ *and* H_n, *namely* $u_n = f_n(H_n, \eta_j,\ j \geqslant 1)$, *with*

$$\frac{u_n}{v_n} \to 1 \text{ in probability,} \tag{3.3}$$

then

$$P\left(\frac{\sum_{j=1}^{u_n} \eta_j}{\sqrt{u_n}} \leqslant x | H_n\right) \to \Phi(x) \text{ in probability} \tag{3.4}$$

for any $x \in \mathbb{R}$, *where* $\Phi(x)$ *is the standard normal distribution function.*

Proof. In a large probability space, we can always define a sequence of independent random variables, say, $\{H_n'\}$, independent of $\{\eta_j\}$, so that for each $n \geqslant 1$, H_n' and H_n have the same distribution. Then by defining $u_n' = f_n(H_n', \eta_j, j \geqslant 1)$ we have

$$P\left(\frac{\sum_{j=1}^{u_n'} \eta_j}{\sqrt{u_n'}} \leqslant x | H_n'\right) \stackrel{\mathrm{d}}{=} P\left(\frac{\sum_{j=1}^{u_n} \eta_j}{\sqrt{u_n}} \leqslant x | H_n\right).$$

Thus, without loss of generality, we can assume that H_n's are independent. Set $\mathscr{F} = \sigma(H_n, n \geqslant 1)$. Then $\{\eta_j\}$ is independent of $\mathscr{F}$. So with probability 1

$$P\left(\frac{\sum_{j=1}^{n} \eta_j}{\sqrt{n}} \leqslant x | \mathscr{F}\right) \to \Phi(x) \quad \text{for all } x \in \mathbb{R}.$$

To prove (3.4), we need only to show that for any given sequence of integers, say, $\{n'\}$, there exists its subsequence along which (3.4) holds.

Now (3.3) implies that for some sequence $\{\delta_n\}$

$$\lim_{n\to\infty} P\left(\left|\frac{u_n}{v_n} - 1\right| > \delta_n\right) = 0 \quad \text{where } \lim_{n\to\infty} \delta_n = 0.$$

Therefore, we have

$$P\left(\left|\frac{u_n}{v_n} - 1\right| > \delta_n | \mathscr{F}\right) \to 0 \text{ in probability.}$$

Now we know that there exists a subsequence, $\{n''\}$, of $\{n'\}$ such that

$$P\left(\left|\frac{u_{n''}}{v_{n''}} - 1\right| > \delta_{n''} | \mathscr{F}\right) \to 0 \quad \text{a.s.}$$

which implies that conditional on $\mathscr{F}$,

$$\frac{u_{n''}}{v_{n''}} \to 1 \text{ in probability.}$$

Since v_n is $\mathscr{F}$-measurable, by applying Theorem 17.1 of Billingsley we have with probability 1,

$$P\left(\frac{\sum_{j=1}^{u_{n''}} \eta_j}{\sqrt{u_{n''}}} \leqslant x | \mathscr{F}\right) \to \Phi(x)$$

for all $x \in \mathbb{R}$. And the lemma follows immediately from the relation

$$P\left(\frac{\sum_{j=1}^{u_n} \eta_j}{\sqrt{u_n}} \leqslant x | H_n\right) = P\left(\frac{\sum_{j=1}^{u_n} \eta_j}{\sqrt{u_n}} \leqslant x | \mathscr{F}\right) \quad \text{a.s.}$$

That completes the proof. □

Proof of Theorem 2.1. In view of (3.1), the almost sure convergence of $\hat{m}_n$ follows from the strong law of large numbers and that $S_n \to \infty$ a.s. as $n \to \infty$. And then the strong convergence of $\hat{m}_G$ follows from that of $\hat{m}_n$ since $G \to \infty$ a.s.

We turn to the proof of the asymptotic normality of $\hat{m}_G$. Note that

$$\hat{m}_G - m = \frac{1}{S_G}\sum_{i=1}^{S_G}(\xi_i - m).$$

For the case $m \leqslant 1$, if we can prove that

$$\frac{S_G}{c\sigma^2} \to 1 \text{ in probability} \tag{3.5}$$

then as in Sriram et al. (1991) we can apply Theorem 17.1 in Billingsley (1968, p. 146) to get (2.7). From Sriram et al. (1991)

$$\frac{S_{N_c}}{c\sigma^2} \to 1 \quad \text{a.s.} \tag{3.6}$$

and from Eq. (2.5) and Lemma 2.2

$$\frac{G}{N_c} \to 1 \quad \text{a.s.}$$

If $m < 1$, then from the ergodic property (see e.g. Wei and Winnicki, 1989, 1990) we have

$$\frac{1}{n} S_n \to \frac{\lambda}{1-m} \quad \text{a.s.}$$

So

$$\frac{S_G}{c\sigma^2} = \frac{1}{\sigma^2}\frac{S_G}{G}\frac{G}{N_c}\frac{N_c}{c} \to 1 \quad \text{a.s.},$$

i.e. (3.5) holds.

In the case $m = 1$, Sriram et al. (1991) showed $N_c/\sqrt{c}$ converges in distribution. Therefore, by Lemma 3.1 and (2.5) we obtain

$$\frac{S_G - S_{N_c}}{c} = \frac{\mathrm{o}(G\log G)(G(c,N_c) - N_c)}{c} = \frac{\mathrm{o}(N_c \log N_c)c^{\alpha}}{c} = \mathrm{o_p}\left(\frac{\log c}{c^{1/2-\alpha}}\right) = \mathrm{o_p}(1)$$

which, together with (3.6), yields (3.5).

Now we focus on the case $m > 1$. By (3.6) it suffices to show

$$\frac{1}{\sigma\sqrt{S_G}}\sum_{i=1}^{S_{N_c}}(\xi_i - m) \to 0 \text{ in probability} \tag{3.7}$$

and

$$\frac{1}{\sigma\sqrt{S_G}}\sum_{i=S_{N_c}+1}^{S_G}(\xi_i - m) \xrightarrow{\mathrm{d}} \mathrm{N}(0,1). \tag{3.8}$$

Let $T_n = \sum_{i=1}^{n}(\xi_i - m)$. It is well known that

$$\frac{1}{\sqrt{n}}\max_{1\leqslant i\leqslant n}|T_i| \text{ converges in distribution.} \tag{3.9}$$

Since $S_{N_c-1} < c\sigma^2 \leqslant S_{N_c}$ and $S_{N_c}/S_{N_c-1} \to m$ a.s. as $c \to \infty$ from (3.2), we have that with probability 1

$$S_{N_c} < [2mc\sigma^2] \text{ ultimately,}$$

where $[x]$ denotes the largest integer less than x. Thus, to show (3.7) it suffices to prove

$$\frac{1}{\sqrt{S_G}} \max_{1 \leqslant i \leqslant [2mc\sigma^2]} |T_i| \to 0 \text{ in probability.}$$

This is trivial by virtue of (3.8) from observing that

$$\frac{S_G}{m^{G-N_c} S_{N_c}} \to 1 \quad \text{a.s.} \tag{3.10}$$

from (2.6) and (3.2) and

$$\liminf_{c\to\infty} \frac{S_G}{[2mc\sigma^2]} = \liminf_{c\to\infty} \frac{S_G}{S_{N_c}} \frac{S_{N_c}}{[2mc\sigma^2]} \geqslant \liminf_{c\to\infty} m^{G-N_c} \frac{c}{[2mc\sigma^2]} = \infty \quad \text{a.s.}$$

from (2.6).

From (3.10),

$$\frac{S_G - S_{N_c}}{m^{G-N_c} S_{N_c}} \to 1 \quad \text{a.s.} \tag{3.11}$$

and hence, proving (3.8) is equivalent to demonstrating

$$\frac{1}{\sigma\sqrt{S_G - S_{N_c}}} \sum_{i=S_{N_c}+1}^{S_G} (\xi_i - m) \xrightarrow{\text{d}} \text{N}(0,1). \tag{3.12}$$

In fact, we need only to prove (3.12) along any given subsequence. We will focus on a given $\{c_n\}$ with $c_n \to \infty$.

Let $H_n = (R_n, N_n, Q_n)$ be distributed as $(S_{N_{c_n}}, N_{c_n}, Z'_{N_{c_n}})$ and independent of $\{\xi_j\}$ and $\{Y_j\}$. Then from Lemma 3.2

$$\frac{1}{\sigma\sqrt{S_{G(c_n,N_{c_n})} - S_{N_{c_n}}}} \sum_{i=S_{N_{c_n}}+1}^{S_G} (\xi_i - m) \stackrel{\text{d}}{=} \frac{1}{\sigma\sqrt{S_{\Delta_n}(Q_n)}} \sum_{i=1}^{S_{\Delta_n}(Q_n)} (\xi_i - m), \tag{3.13}$$

where $\Delta_n := G(c_n, N_n) - N_n$ depends only on c_n and N_n.

Since $S_{\Delta_n}(Q_n)$ and $S_G - S_{N_c}$ have the same distribution, we have from (3.11) that

$$\frac{S_{\Delta_n}(Q_n)}{m^{\Delta_n} R_n} \to 1 \text{ in probability.}$$

Thus, in view of Lemmas 3.2 and 3.3, (3.12) is proved. □

4. Simulations

In this section we report some simulation results.

In our simulations we assume that the offspring follows a geometric (p) distribution. Note the mean $m = (1-p)/p$ and the variance $\sigma^2 = (1-p)/p^2$. Thus, $m < 1$, $m = 1$

Table 1
Subcritical case ($p = 0.8$, $m = 0.25$, $\sigma^2 = 0.3125$)

c	Ave of $\hat{m}$	RMSE of $\hat{m}$	Ave of G
100	0.248196	0.0867666	11.4637
200	0.249163	0.0654826	19.4177
300	0.250260	0.0543926	28.0401
400	0.249691	0.0472902	35.8652
500	0.250087	0.0427055	43.8015

Table 2
Critical case ($p = 0.5$, $m = 1$, $\sigma^2 = 2$)

c	Ave of $\hat{m}$	RMSE of $\hat{m}$	Ave of G
100	0.991870	0.0809523	15.1897
200	0.995480	0.0602316	20.6430
300	0.995999	0.0501390	24.8959
400	0.996531	0.0439171	28.6094
500	0.997357	0.0392181	31.6325

Table 3
Supercritical case ($p = 0.2$, $m = 4$, $\sigma^2 = 20$)

c	Ave of $\hat{m}$	RMSE of $\hat{m}$	Ave of G
100	4.000039	0.0360439	7.5178
200	4.000292	0.0249652	8.1392
300	4.000068	0.0186907	8.5771
400	4.000109	0.0158365	8.9125
500	3.999994	0.0135867	9.1645

or $m > 1$ according as $p > 0.5$, $p = 0.5$ or $p < 0.5$. The immigration is assumed to have a uniform distribution over $\{1, 2, 3, 4, 5\}$. The initial population Z_0 is set as 1.

We simulated the data by selecting p from 0.2 to 0.8 with increments of 0.1 and c from 100 to 500 with increments of 100. Since these results are quite similar, we report only three cases with $p = 0.2$, 0.5 and 0.8, corresponding to the means $m = 4$, 1 and 0.25, which cover the supercritical, critical and subcritical cases (Tables 1–3).

Here, we choose $G(c, N_c) = g(N_c) + N_c$, where $g(x) = [\log(x)]$, the integer part of $\log(x)$. Then it is easily checked that both (2.5) and (2.6) hold.

The tables include the information:

$$\text{Ave of } \hat{m} = \text{average of } \hat{m}_G \text{ from } 10{,}000 \text{ simulations},$$

$$\text{RMSE of } \hat{m} = \text{root of mean squared error of } \hat{m}_G = \sqrt{\frac{1}{10{,}000} \sum (\hat{m}_G - m)^2}$$

and

$$\text{Ave of } G = \text{average of } G \text{ from } 10{,}000 \text{ simulations}.$$

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