Chover’s Law of the Iterated Logarithm for Trimmed Sums

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Abstract

Consider a sequence of independent and identically distributed random variables with the underlying distribution in the domain of attraction of a stable distribution with an exponent in $(0, 2]$. Define the trimmed sums as the partial sums excluding $r$ largest observations in magnitude, where $r$ is a fixed integer. This paper proves that Chover’s law of the iterated logarithm holds for the trimmed sums.


Keywords and phrases. Trimmed sum, law of the iterated logarithm, stable law, domain of attraction.

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with a non-degenerate distribution function $F$. For each $n \geq 1$, set $S_n = \sum_{j=1}^{n} X_j$. For integer $r$ with $1 \leq r \leq n$, let $X_n^{(r)} = X_m$ if $|X_m|$ is the $r$-th maximum of $\{|X_k|; 1 \leq k \leq n\}$. Then $(r)S_n = S_n - (X_n^{(1)} + \cdots + X_n^{(r)})$ is called trimmed sum. When $r$ is fixed, $(r)S_n$ is called lightly trimmed sum. For $r = 0$, $(0)S_n$ is defined as the partial sum $S_n$. In this paper, we consider lightly trimmed sums only.

Strong limit laws and asymptotic distributions of the lightly trimmed sums have been studied by several authors, see, e.g., Mori (1976, 1977, 1984) and Maller (1982, 1984). It is well known that light trimming does not improve weak convergence results but does improve strong convergence of the partial sum $S_n$. 
Assume that $F$ is in the domain of attraction of a stable distribution $G_\alpha$ with exponent $\alpha \in (0, 2]$, i.e., there exist some constants $A_n \in R, B_n > 0$ such that

$$\frac{S_n - A_n}{B_n} \xrightarrow{d} G_\alpha. \quad (1.1)$$

It is well known that (1.1) holds with $\alpha \in (0, 2)$ if and only if

$$1 - F(x) = \frac{C_1(x)l(x)}{x^\alpha} \quad \text{and} \quad F(-x) = \frac{C_2(x)l(x)}{x^\alpha} \quad \text{for } x > 0, \quad (1.2)$$

where, for $x > 0$, $C_i(x) \geq 0, \lim_{x \to \infty} C_i(x) = C_i, i = 1, 2, C_1 + C_2 > 0$, and $l(x) \geq 0$ is a slowly varying function, i.e.,

$$\lim_{t \to \infty} \frac{l(tx)}{l(t)} = 1 \quad \text{for } x > 0.$$

It is also known that (1.1) holds with $\alpha = 2$, i.e., $F$ is in the domain of attraction of the normal distribution if and only if

$$\lim_{x \to \infty} \frac{x^2P(|X_1| > x)}{E\{X_1^2I(|X_1| \leq x)\}} = 0. \quad (1.3)$$

When $X$ has a symmetric stable distribution function $F$ characterized by

$$E\{\exp(itX)\} = \exp(-|t|^\alpha) \quad \text{for } t \in R, \quad (1.4)$$

Chover (1966) established that

$$\limsup_{n \to \infty} \left| \frac{S_n}{n^{1/\alpha}} \right|^{\frac{1}{\log \log n}} = e^{1/\alpha} \quad \text{a.s.,}$$

which is known as Chover’s law of the iterated logarithm (LIL). Under general assumption (1.1), Qi and Cheng (1996) proved that

$$\limsup_{n \to \infty} \left| \frac{1}{B_n} (S_n - A_n) \right|^{\frac{1}{\log \log n}} = e^{1/\alpha} \quad \text{a.s. \quad if } \alpha \in (0, 2), \quad (1.5)$$

and

$$\limsup_{n \to \infty} \left| \frac{1}{B_n} (S_n - A_n) \right|^{\frac{1}{\log \log n}} = : \beta \in [1, e^{1/2}] \quad \text{a.s. \quad if } \alpha = 2. \quad (1.6)$$

Vasudeva (1984) also obtained a similar result under assumption (1.1) with $A_n \equiv 0$ for $\alpha \in (0, 2)$. Some recent papers have been devoted to the study
of Chover’s LIL for weighted sums. We refer the reader to Chen and Huang (2000), Chen (2002), Chen and Liu (2003), Peng and Qi (2003) and Chen and Qi (2006).

In this paper, we will prove Chover’s LIL for the lightly trimmed sum $\sum^{(r)} S_n$. We will state our main results and some examples in Section 2 and give the proofs in Section 3.

2 Main Results

Set $G(x) = P(|X_1| \geq x)$, $K(x) = 2x^{-2} \int_0^x tG(t)dt$, and define

$$B(x) = \inf \left\{ y : G(y) \leq \frac{1}{x} \right\} \quad \text{for } x > 0,$$

$$D(x) = \inf \left\{ y > 0 : K(y) \leq \frac{1}{x} \right\}, \quad \text{for } x > 0.$$

Set $\log_0 x = x$ and denote $\log_j x = \log\{\max(e, \log_{j-1} x)\}$ for $j \geq 1$. We also define $\log x = \log_1 x$.

Throughout this paper, we assume $r, k \geq 0$ are fixed integers. For convenience, the product $\prod_{j=1}^k \cdot$ is defined as 1 if $k < 1$.

**Theorem 2.1.** Assume that (1.1) holds with some $\alpha \in (0, 2)$. Then

$$\limsup_{n \to \infty} \left( \frac{|\sum^{(r)} S_n - A_n|}{B(n/\prod_{j=1}^k \log_j n)^{1/(r+1)}} \right)^{1/\log_{k+2} n} = e^{a/(r+1)} \quad \text{a.s.},$$

where $\{A_n\}$ is any sequence that satisfies (1.1).

**Remark.** In Theorem 2.1, if we set $k = 0$, then we obtain Chover’s LIL for the trimmed sums

$$\limsup_{n \to \infty} \left( \frac{|\sum^{(r)} S_n - A_n|}{B(n)} \right)^{1/\log_2 n} = e^{a/(r+1)} \quad \text{a.s.}.$$

It is worth mentioning that $B(n)$ satisfies (1.1) and can be replaced by $B_n$ since there exists a constant $c > 0$ such that

$$\frac{B_n}{B(n)} \to c \quad \text{if } \alpha \in (0, 2),$$
and
\[
\frac{B_n}{D(n)} \to c \quad \text{if } \alpha = 2 \tag{2.1}
\]
from Convergence of Types Theorem (see, e.g., Loève, 1977). In particular, if we choose \( r = 0 \), we obtain Chover’s LIL for the partial sum, i.e., (1.5) and (1.6).

**Theorem 2.2.** Assume (1.1) holds with \( \alpha = 2 \). Then, there exists a constant \( \beta \) such that
\[
\limsup_{n \to \infty} \left( \frac{|(r)S_n - A_n|}{B_n} \right)^{1/\log_2 n} = \beta \in [1, e^{1/(r+1)}] \quad \text{a.s.,} \tag{2.2}
\]
where \( \{A_n\} \) and \( \{B_n\} \) are any sequences satisfying (1.1).

It is natural to ask, for any given \( \beta \in [1, e^{1/(r+1)}] \), whether there exists a distribution \( F \) satisfying (1.3) such that (2.2) holds. Our answer is positive. Note that under (1.3),
\[
EX^2 I(|X_1| \leq x) \sim 2 \int_0^x tG(t)dt \quad \text{as } x \to \infty,
\]
and thus we have
\[
R(x) =: \frac{G(x)}{K(x)} = \frac{x^2 G(x)}{2 \int_0^x tG(t)dt} \to 0 \tag{2.3}
\]
as \( x \to \infty \).

**Theorem 2.3.** Assume that (1.1) holds with \( \alpha = 2 \), and \( \delta \in [0, 1] \) is fixed. Suppose that \( R(x) \), defined as in (2.3), is such that, for every \( \varepsilon \in (0, 1) \),
\[
\lim_{x \to \infty} R(x)(\log x)^{(\delta + \varepsilon)/(r+1)} = \infty, \quad \lim_{x \to \infty} R(x)(\log x)^{(\delta - \varepsilon)/(r+1)} = 0 \tag{2.4}
\]
when \( \delta \in (0, 1) \),
\[
\lim_{x \to \infty} R(x)(\log x)^{(1-\varepsilon)/(r+1)} = 0
\]
when \( \delta = 1 \), and
\[
\lim_{x \to \infty} R(x)(\log x)^\varepsilon = \infty
\]
when \( \delta = 0 \). Then
\[
\limsup_{n \to \infty} \left( \frac{|(r)S_n - A_n|}{B_n} \right)^{1/\log_2 n} = e^{\frac{1-\delta}{2(r+1)}} \quad \text{a.s..} \tag{2.5}
\]
We end this section with an example that shows that for each \( \delta \in [0, 1] \), there exists a distribution \( F \) such that conditions in Theorem 2.3 are satisfied.

**Example.** Let \( r \geq 0 \) be a fixed integer and \( \delta \in [0, 1] \). Assume that \( X_1 \) has a symmetric distribution with

\[
G(x) = P(|X_1| > x) = x^{-2} \exp \left\{ \int_1^x \frac{2}{t(\log t)^{\delta/(r+1)} \log x} dt \right\}, \quad x > 1.
\]

It is easy to check that

\[
2 \int_0^x tG(t) dt \sim \exp \left\{ \int_1^x \frac{2}{t(\log t)^{\delta/(r+1)} \log x} dt \right\} (\log x)^{\delta/(r+1)} \log x \quad \text{as} \quad x \to \infty,
\]

which implies that

\[
R(x) \sim \frac{1}{(\log x)^{\delta/(r+1)} \log x}.
\]

Then the conditions of Theorem 2.3 are satisfied. Hence (2.5) holds.

### 3 Proofs

We start with some lemmas.

**Lemma 3.1.** Suppose that \( h(x) \) is a slowly varying function at infinity and \( g(x) \) is a positive function with \( \lim_{x \to \infty} g(x) = \infty \). Then, for any given \( \delta > 0 \), there exists an \( x_0 > 0 \) such that

\[
g^{-\delta}(x) < \inf_{x \leq y \leq xg(x)} \frac{h(y)}{h(x)} \leq \sup_{x \leq y \leq xg(x)} \frac{h(y)}{h(x)} < g^\delta(x) \quad \text{for all} \quad x > x_0.
\]

Hence, \( \log (h(xg(x))/h(x)) = o(\log g(x)) \) as \( x \to \infty \).

See Lemma 1 in Qi and Cheng (1996) for proof.

**Lemma 3.2.** Assume that \( \{X_n, n \geq 1\} \) is a sequence of i.i.d. random variables, and \( \{a_n\} \) is a sequence of positive constants satisfying

(a) \( \{a_n/n^{1/\gamma}\} \) is nondecreasing ultimately for some \( \gamma \in (0, 2) \);

(b) \( \sup_n \{a_{2n}/a_n\} < \infty \).

If \( \sum_n n^r Pr^r(|X_1| > a_n) < \infty \), then

\[
\lim_{n \to \infty} \left( r \right) S_n - \frac{nE \{X_1 I(|X_1| \leq a_n)\}}{a_n} = 0 \quad \text{a.s.}
\]
If \( \sum_n n^r P^{r+1}(|X_1| > a_n) = \infty \), then
\[
\limsup_{n \to \infty} \frac{|(r)S_n - c_n|}{a_n} = \infty \text{ a.s.}
\]
for every sequence \( \{c_n\} \).

See Mori (1977) (Theorem 1 and Lemmas 1 and 2) for proof.

**Lemma 3.3.** Assume that \( \{X_n, n \geq 1\} \) is a sequence of i.i.d. random variables, and \( \{a_n\} \) is a non-decreasing sequence of positive constants with \( \sup_n \frac{a_{n+1}}{a_n} < \infty \) and \( \lim_{n \to \infty} a_n = \infty \). If
\[
\sum_n n^r P^{r+1}(|X_1| > a_n) = \infty,
\]
then
\[
\limsup_{n \to \infty} \frac{|(r)S_n - c_n|}{a_n} > 0 \text{ a.s.}
\]
for every sequence \( \{c_n\} \).

**Proof.** Otherwise, from zero-one law, there exists some sequence \( \{c_n\} \) such that
\[
\limsup_{n \to \infty} \frac{|(r)S_n - c_n|}{a_n} = 0 \text{ a.s.}
\]
Hence,
\[
\limsup_{n \to \infty} \left| \frac{(r)S_{n+1} - (r)S_n}{a_n} - \frac{c_{n+1} - c_n}{a_n} \right| = 0 \text{ a.s.} \quad (3.1)
\]
By noting that
\[
\frac{|(r)S_{n+1} - (r)S_n|}{a_n} = \min \left( \frac{|X_{n+1}|}{a_n}, \frac{|X_{n}^{(r)}|}{a_n} \right), \quad (3.2)
\]
which converges in distribution to zero, we conclude from (3.1) that
\[
\limsup_{n \to \infty} \left| \frac{c_{n+1} - c_n}{a_n} \right| = 0,
\]
which coupled with (3.1) yields
\[
\limsup_{n \to \infty} \frac{|(r)S_{n+1} - (r)S_n|}{a_n} = 0 \text{ a.s.} \quad (3.3)
\]
Using the fact that 
\[ |X_{n+1}^{(r+1)}| \leq \max_{r \leq m \leq n} \min(|X_{m+1}|, |X_m^r|), \]
(cf. Kuelbs and Ledoux, 1987), we have from (3.2) and (3.3) that
\[ \limsup_{n \to \infty} \frac{|X_{n+1}^{(r+1)}|}{a_n} = 0 \text{ a.s.}, \]
which implies, by Lemma 3 in Mori (1976), that
\[ \sum_{n=1}^{\infty} n^r P^{r+1}(|X_1| > a_n) < \infty. \]
This is contradictory to the assumption of the lemma. Hence the lemma is proved. \(\square\)

**Lemma 3.4.** Assume that \(a(x)\) is a regularly varying function and \(a_n = a(n)\) for \(n \geq 1\). If
\[ \sum_{n=1}^{\infty} n^r P^{r+1}(|X_1| > a_n) < \infty, \]
then there exists a sequence of independent standard normal random variables \(\{Y_n, n \geq 1\}\) (in a larger probability space) such that
\[ (r)S_n - nE\{X_1 I(|X_1| \leq a_n)\} - \sum_{j=1}^{n} \sigma_j Y_j = o(a_n) \text{ a.s.,} \]
where \(\sigma_n^2 = \text{Var}\{X_1 I(|X_1| \leq a_n)\}\) for \(n \geq 1\).

See, e.g., Theorem 3 of Zhang (2003).

**Proof of Theorem 2.1.** From (1.2), \(G(x) = P(|X_1| > x)\) is a regularly varying function with index \(-\alpha\) at infinity. Hence, from de Haan (1970), \(B(x)\) is a regularly varying function with index \(1/\alpha\) at infinity, and by Karamata’s representation,
\[ B(x) = c_1(x)x^{1/\alpha} \exp \left\{ \int_1^x \frac{b_1(u)}{u} du \right\}, \]
where \(\lim_{x \to \infty} c_1(x) = c_1 \in (0, \infty)\) and \(\lim_{x \to \infty} b_1(x) = 0\). Using the properties of regular variation (see Seneta, 1976), we have
\[ \lim_{x \to \infty} \frac{\lg B(x)}{\lg x} = 1/\alpha, \quad \lim_{x \to \infty} \frac{\lg_2 B(x)}{\lg_2 x} = 1 \] \(\text{(3.4)}\)
and

\[ \lim_{x \to \infty} x G(B(x)) = 1. \quad (3.5) \]

Set now \( b(x) = c_1 x^{1/\alpha} \exp\{ \int_1^x \frac{b_1(u)}{u} du \} \). It is easy to see that

\[ B(x) \sim b(x) \quad \text{as} \quad x \to \infty, \quad (3.6) \]

and that for any given \( \gamma \in (\alpha, 2) \), there exists an \( x_\gamma > 0 \) such that \( b(x)/x^{1/\gamma} \) is increasing in \( (x_\gamma, \infty) \). Then for any fixed \( \delta > 0 \), the conditions of Lemma 3.2 are satisfied by the sequence \( \{a_n = b \left( n \left( \prod_{j=1}^k \lg n \right)^{(1/\gamma)/\alpha(r+1)} \right) \} \).

Moreover, from (3.5) and (3.6), we have

\[ G \left( b \left( n \left( \prod_{j=1}^k \lg n \right)^{(1/\gamma)/\alpha(r+1)} \right) \right) \sim \frac{1}{n \left( \prod_{j=1}^k \lg n \right)^{(1/\gamma)/\alpha(r+1)}} \]

for all \( \delta > 0 \). Therefore, we obtain

\[ \sum_n n^{r+1} \mathbb{P} \left( \left| X_1 \right| > b \left( n \left( \prod_{j=1}^k \lg n \right)^{(1/\gamma)/\alpha(r+1)} \right) \right) < \infty \]

for all \( \varepsilon > 0 \), and

\[ \sum_n n^{r+1} \mathbb{P} \left( \left| X_1 \right| > b \left( n \left( \prod_{j=1}^k \lg n \right)^{(1-\varepsilon)/\alpha(r+1)} \right) \right) = \infty \]

for all \( \varepsilon \in (0, 1) \). By using (3.7), we have from Lemma 3.2 that

\[ \lim_{n \to \infty} \frac{(r)S_n - nE \{ X_1 I(\left| X_1 \right| \leq a_n) \}}{a_n} = 0 \quad \text{a.s.} \quad (3.9) \]

for every \( \varepsilon \in (0, 1) \), where \( a_n = b \left( n \left( \prod_{j=1}^k \lg n \right)^{(1+\varepsilon)/\alpha(r+1)} \right) \).

Since \( \left| X_n^{(1)} \right| = \max_{1 \leq j \leq n} \left| X_j \right| \), from the choice of \( B_n \), one can see that the sequence \( \{|X_n^{(1)}|/B_n\} \) is tight. Hence, we have

\[ \left| \frac{(r)S_n - A_n}{a_n} \right| \leq B_n \left( \frac{r \left| X_n^{(1)} \right|}{B_n} + \frac{|S_n - A_n|}{B_n} \right). \]

This converges to zero in probability, which coupled with (3.9) yields that

\[ \frac{A_n - nE \{ X_1 I(\left| X_1 \right| \leq a_n) \}}{a_n} \to 0 \quad \text{as} \quad n \to \infty. \]
Therefore, from (3.9) and (3.6), we conclude
\[
\lim_{n \to \infty} \frac{(r) S_n - A_n}{a'_n} = 0 \quad \text{a.s.,}
\]
where \( a'_n = B \left( n \left( \prod_{j=1}^{k} \log j n \right)^{1/(r+1)} \right) \cdot \left( \log k + 1 \right)^{(1+\varepsilon)/\alpha(r+1)} \). Let “i.o.” denote “infinitely often”. Then, we have
\[
P(|(r) S_n - A_n| > a'_n \quad \text{i.o.}) = 0,
\]
or equivalently
\[
P \left( \left( \frac{|(r) S_n - A_n|}{B \left( n \left( \prod_{j=1}^{k} \log j n \right)^{1/(r+1)} \right)} \right)^{1/\log n} \right) \left( \frac{a'_n}{\left( \prod_{j=1}^{k} \log j n \right)^{1/(r+1)}} \right)^{1/\log n} = 0. \quad (3.10)
\]
Rewrite \( B(x) \) as \( B(x) = x^{1/\alpha} L(x) \), where \( L(x) \) is a slowly varying function at infinity. Then, from Lemma 3.1,
\[
\frac{a'_n}{\left( \prod_{j=1}^{k} \log j n \right)^{1/(r+1)}} = \left( \log k + 1 \right)^{(1+\varepsilon)/\alpha(r+1)} \cdot \frac{L \left( n \left( \prod_{j=1}^{k} \log j n \right)^{1/(r+1)} \right)}{L \left( \left( \prod_{j=1}^{k} \log j n \right)^{1/(r+1)} \right)} \leq \left( \log k + 1 \right)^{(1+\varepsilon)/\alpha(r+1)} \cdot \left( \log k + 1 \right) \varepsilon \alpha(r+1) \leq \left( \log k + 1 \right)^{(1+2\varepsilon)/\alpha(r+1)}
\]
holds for all large \( n \). This, together with (3.10), implies that
\[
\limsup_{n \to \infty} \left( \frac{|(r) S_n - A_n|}{B \left( n \left( \prod_{j=1}^{k} \log j n \right)^{1/(r+1)} \right)} \right)^{1/\log n} \leq e^{(1+2\varepsilon)/\alpha(r+1)} \quad \text{a.s.}
\]
Since \( \varepsilon > 0 \) is arbitrary, we get the upper bound
\[
\limsup_{n \to \infty} \left( \frac{|(r) S_n - A_n|}{B \left( n \left( \prod_{j=1}^{k} \log j n \right)^{1/(r+1)} \right)} \right)^{1/\log n} \leq \left( \frac{\alpha(r+1)}{\varepsilon} \right) \quad \text{a.s.} \quad (3.11)
\]
A lower bound

\[
\limsup_{n \to \infty} \left( \frac{|(r)S_n - A_n|}{B \left( n \left( \prod_{j=1}^{k} \log n \right)^{1/(r+1)} \right)} \right)^{1/\log k + 2} n \geq e^{1/\alpha(r+1)} \quad \text{a.s.} \quad (3.12)
\]

can be obtained by using (3.8) and Lemma 3.2 in a similar fashion.

The proof of Theorem 2.1 is complete in view of (3.11) and (3.12).

**Proof of Theorem 2.2.** From (2.1), it suffices to show (2.2) with \(B_n\) replaced by \(D(n)\). From zero-one law, a non-random constant limit \(\beta \in [0, \infty]\) in (2.2) exists. Hence, it suffices to show that this limit \(\beta \in [1, e^{1/2(r+1)}]\). If \(EX_1^2 < \infty\), it follows trivially from Hartman-Wintner's law of the iterated logarithm that \(\beta = 1\). Hence we assume \(EX_1^2 = \infty\) below.

Set \(H(x) = 2 \int_0^x tG(t)dt\). Then \(K(x) = H(x)/x^2\). From (1.1), we have

\[
\frac{S_n - A_n}{D(n)} \xrightarrow{d} N(\mu, \sigma^2), \quad (3.13)
\]

where \(\mu \in \mathbb{R}\) and \(\sigma^2 > 0\), which implies

\[
\frac{(r)S_n - A_n}{D(n)} \xrightarrow{d} N(\mu, \sigma^2),
\]

since

\[
\frac{|X_n^{(1)}|}{D(n)} \xrightarrow{p} 0
\]

when \(X_1\) is in the domain of attraction of a normal distribution. Set \(\varepsilon_n = 1/\log 2n\). Then,

\[
P \left( \frac{|(r)S_n - A_n|}{D(n)} > \varepsilon_n \right) \to 1.
\]

Therefore,

\[
P \left( \frac{|(r)S_n - A_n|}{D(n)} > \varepsilon_n \text{ i.o.} \right) = \lim_{j \to \infty} P \left( \bigcup_{n=j}^{\infty} \left\{ \frac{|(r)S_n - A_n|}{D(n)} > \varepsilon_n \right\} \right) \geq \lim_{j \to \infty} P \left( \frac{|(r)S_j - A_j|}{D(j)} > \varepsilon_j \right) = 1,
\]
from which we conclude that

$$\beta = \limsup_{n \to \infty} \left( \frac{|(r)S_n - An|}{D(n)} \right)^{1/\lg n} \geq \lim_{n \to \infty} (\varepsilon_n)^{1/\lg n} = 1.$$ 

Thus, what remains is to prove that $\beta \leq e^{1/2(r+1)}$.

From (2.3), one obtains

$$H(x) = H(1) \exp \left\{ \int_1^x \frac{2E(t)}{t} dt \right\} \quad \text{for } x > 1.$$ 

Thus, $H(x)$ is a slowly varying function at infinity, and $D(x)$ is regularly varying with index $1/2$, i.e.,

$$D(x) = x^{\frac{1}{2}}q(x), \quad (3.14)$$

where $q(x)$ is slowly varying (see de Haan, 1970, p.22). By virtue of the properties of regular variation, we have

$$\lim_{x \to \infty} xK(D(x)) = 1.$$ 

Set $a(x) = D(x(\lg x)^{1/(r+1)}(\lg x)^{2/(r+1)})$ and $a_n = a(n)$. From (2.3), we have

$$P(|X_1| > a(x)) = G(a(x)) = o \left( K(a(x)) \right) = o \left( \frac{1}{x(\lg x)^{1/(r+1)}} \right),$$

which implies $\sum_n n^rP^{r+1}(|X_1| > a_n) < \infty$. From Lemma 3.4, we obtain

$$(r)S_n - nE\{X_1I(|X_1| \leq a_n)\} - \sum_{j=1}^n \sigma_j Y_j = o(a_n) \quad \text{a.s.} \quad (3.15)$$

Set $T_n = \sum_{j=1}^n \sigma_j Y_j$ and $\tau_n^2 = Var(T_n)$ for $n \geq 1$. Since $EX_1^2 = \infty$, we have

$$\sigma_j^2 \sim E \{ X_1^2 I(|X_1| \leq a_j) \} \sim H(a_j) \quad \text{as } j \to \infty. \quad \text{Therefore,}$$

$$\tau_n^2 = \sum_{j=1}^n \sigma_j^2 \sim \sum_{j=1}^n H(a_j) \sim \int_0^n H(a(x))dx.$$ 

Since $H(x)$ is slowly varying and $a(x)$ is regularly varying, $H(a(x))$ is slowly varying. From Karamata’s theorem (cf. Bingham et al., 1987, Proposition
1.5.8), we have \( \int_0^n H(a(x)) dx \sim nH(a(n)) \). Hence,

\[
\tau_n^2 \sim nH(a(n)) = na_n^2 K \left( D \left( n(\log n)^{1/(r+1)}(\log_2 n)^{2/(r+1)} \right) \right) \\
\sim \frac{a_n^2}{(\log n)^{1/(r+1)}(\log_2 n)^{2/(r+1)}},
\]

which implies \( T_n/a_n \xrightarrow{p} 0 \) from Markov inequality. Since \( D(n) = o(a_n) \), we conclude from (3.13) and (3.15) that \( [A_n - nE\{X_1 I(|X_1| \leq a_n)\}]/a_n \rightarrow 0 \) as \( n \rightarrow \infty \). Once again, from (3.15), we get

\[
(r)S_n - A_n - T_n = o(a_n) \quad \text{a.s.} \tag{3.16}
\]

Note that \( T_n \) is a weighted sum of normal random variables. Since \( \tau_n^2 / \tau_n^2 \rightarrow 1 \) as \( n \rightarrow \infty \), it follows from Theorem 7.2 in Petrov (1995) that

\[
\limsup_{n \rightarrow \infty} \frac{|T_n|}{\tau_n^2 |\log_2 (\tau_n^2)|} = \sqrt{2} \quad \text{a.s.}
\]

It is trivial to see that \( \log_2 \tau_n^2 \sim \log_2 n \) as \( n \rightarrow \infty \). We conclude from (3.16) that \( T_n = o(a_n) \) a.s. and

\[
\limsup_{n \rightarrow \infty} \left( \frac{|(r)S_n - A_n|}{a_n} \right)^{1/\log_2 n} \leq 1 \quad \text{a.s.} \tag{3.17}
\]

Finally, from Lemma 3.1 and equation (3.14), we conclude that

\[
\lim_{n \rightarrow \infty} \left( \frac{a_n}{D(n)} \right)^{1/\log_2 n} = \lim_{n \rightarrow \infty} \left( \frac{D(n)(\log n)^{1/(r+1)}(\log_2 n)^{2/(r+1)}}{D(n)} \right)^{1/\log_2 n} = e^{1/2(r+1)},
\]

which together with (3.17) yields

\[
\beta = \limsup_{n \rightarrow \infty} \left( \frac{|(r)S_n - A_n|}{D(n)} \right)^{1/\log_2 n} \leq e^{1/2(r+1)}.
\]

This completes the proof of the theorem. \( \square \)
Proof of Theorem 2.3. We prove the theorem when \( \delta \in (0, 1) \) only. The proof is similar when \( \delta = 1 \) or \( \delta = 0 \).

Assume (2.4) holds. It suffices to show that for every small \( \varepsilon \in (0, 1) \),

\[
\limsup_{n \to \infty} \left( \frac{|(r)S_n - A_n|}{D(n)} \right)^{1/\log_2 n} \geq e^{\frac{1-\delta-\varepsilon}{2(r+1)}} \text{ a.s.} \tag{3.18}
\]

and

\[
\limsup_{n \to \infty} \left( \frac{|(r)S_n - A_n|}{D(n)} \right)^{1/\log_2 n} \leq e^{\frac{1-\delta+\varepsilon}{2(r+1)}} \text{ a.s.} \tag{3.19}
\]

From (2.3), we have \( G(x) = R(x)K(x) \) for \( x > 0 \).

Set \( a_n = D \left( n \log n \right)^{(1-\delta-\varepsilon)/(r+1)} \) for \( n \geq 1 \). Then from Lemma 3.1 and (3.14), we have

\[
\lim_{n \to \infty} \left( \frac{a_n}{D(n)} \right)^{1/\log_2 n} = \lim_{n \to \infty} \left( \frac{D(n \log n)^{(1-\delta-\varepsilon)/(r+1)}}{D(n)} \right)^{1/\log_2 n} = e^{\frac{1-\delta-\varepsilon}{2(r+1)}}. \tag{3.20}
\]

Note that

\[
P(|X_1| > a_n) = G(a_n) = R(a_n)K \left( D \left( n \log n \right)^{(1-\delta-\varepsilon)/(r+1)} \right) \geq \frac{(\log n)^{-(\delta+\varepsilon)/(r+1)}}{n \log n^{(1-\delta-\varepsilon)/(r+1)}} = \frac{1}{n \log n^{1/(r+1)}}
\]

for all large \( n \). Therefore, we have \( \sum_n n^rP^{r+1}(|X_1| > a_n) = \infty \). From Lemma 3.3,

\[
\limsup_{n \to \infty} \left( \frac{|(r)S_n - A_n|}{a_n} \right)^{1/\log_2 n} \geq 1 \text{ a.s.,}
\]

which coupled with (3.20) yields (3.18).

In order to prove (3.19), set \( a_n = D \left( n \log n \right)^{(1-\delta+\varepsilon)/(r+1)} \left( \log_2 n \right)^2 \) for \( n \geq 1 \). From (2.4), it is easy to conclude that \( \sum_n n^rP^{r+1}(|X_1| > a_n) < \infty \). By using Lemma 3.4 and the same procedure as in the proof of Theorem 2.2, we can conclude (3.19). The detail is omitted here. \( \square \)
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